A preventive maintenance policy for a bivariate wear indicator with continuous monitoring

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Abstract:

A continuously monitored system is considered, that gradually and stochastically deteriorates according to a bivariate non decreasing Lévy process. The system is considered as failed as soon as the bivariate indicator enters a failure region $\mathcal{L}$. The point of the paper is to study a preventive maintenance policy for the system, which aims at reducing a cost function on some infinite horizon time.

A first point for the study is to propose a model for the bivariate increasing deterioration. Classical univariate (cumulative) wear processes are Gamma and compound Poisson processes, which both are univariate subordinators (increasing Lévy processes). We hence propose to use a bivariate subordinator as bivariate degradation indicator, with univariate subordinators as marginal processes.

Considering a system with deterioration modelled by a bivariate subordinator and failure region $\mathcal{L}$ which is an upper set. It ensures that once the system fails, it remains in failure state without repair. We add the following assumptions: the system is continuously monitored and at failure, a repair team is immediately called, which arrives after a delay τ. A perfect and instantaneous repair is next performed in the failure time plus τ. The future evolution of the system after time τ is an exact and independent stochastic replica of the initial system evolution. A cost function is then used to measure the system performance on an infinite horizon time, which includes some replacement cost and unitary down-time cost.

A preventive maintenance (PM) policy is next proposed for the system, where the repair team is now called when the system enters a preventive region $\mathcal{M}$ (an upper set which includes $\mathcal{L}$).

The points of the study are: first, compute the asymptotic cost of the system submitted to this PM policy; study the influence of the different PM parameters on the cost function; study the influence of the dependence between the marginal wear indicators and of the failure and PM regions on the cost function.

Keywords: Multivariate Lévy processes; dependent wear indicators; optimal replacement; renewal theory.

1. INTRODUCTION

In case of a system submitted to an accumulative random damage, classical stochastic models are compound Poisson processes and Gamma processes, which both are increasing Lévy processes,see [1], [9] or [10] e.g. Such classical wear models typically are univariate. However, the deterioration level of a system cannot always be synthetized into one single indicator and several indicators may be necessary, see [6] for an industrial example. In that case, a multivariate wear model must be used to account for the dependence between the different univariate indicators of the system. Another context where multivariate wear models are required is the case of different systems submitted to common stresses, which make their wear indicators dependent. Multivariate increasing stochastic models hence are of interest in different contexts.We here propose to use multivariate increasing Lévy processes (or multivariate subordinators) as wear processes.

Under such an assumption, a system is considered, subject to continuous monitoring. It is considered as failed as soon as its bivariate deterioration level has reached a failure zone $\mathcal{L}$. Once in $\mathcal{L}$, the system cannot leave $\mathcal{L}$ without being repaired. This property is translated through the assumption that $\mathcal{L}$ is an upper set. As in [3], when the system enters $\mathcal{L}$, a signal is immediately sent to a repair team. It takes some delay τ for the repair team to arrive. The repair duration is short compared to the delay τ and it is hence considered as instantaneous (and perfect). To shorten the system down-time, a preventive maintenance (PM) policy is proposed, where the signal is sent to the repair team as soon as the deterioration level reaches a PM zone $\mathcal{M}$, larger than $\mathcal{L}$.

The paper is organized as follows: in Section 2, the model is presented, both for the initial (without maintenance) and preventively maintained system. Section 3 is devoted to theoretical developments whereas Section 4 presents some numerical experiments. We finally conclude in Section 5.

2. THE MODEL

2.1. The initial system

A system is considered with deterioration level measured by a bivariate non decreasing process $\left(X_t = \left(X_t^{(1)}, X_t^{(2)}\right)\right)_{t\geq 0}$. The process $(X_t)_{t\geq 0}$ is assumed to be a bivariate subordinator with null drift, namely a pure jump process. To avoid trivialities, the process $(X_t)_{t\geq 0}$ is also assumed to be non zero: $\mathbb{P}\left(X_t^{(1)} > 0, X_t^{(2)} > 0\right) > 0$. Such assumptions will be referred to as assumption $\mathcal{H}$. For $i = 1, 2$, the marginal process $\left(X_t^{(i)}\right)_{t\geq 0}$ is known to be an univariate subordinator (with null drift). The system is continuously and perfectly monitored. It is considered as failed as soon as its bivariate deterioration level reaches a failure zone $\mathcal{L} \subset \mathbb{R}_+^2$. The failure time of the unmaintained system hence is:

$$\sigma_{\mathcal{L}} = \inf\{t \geq 0 | X_t \in \mathcal{L}\}.$$

As explained in the introduction, $\mathcal{L}$ is assumed to be a closed and non empty upper set, namely such that for all $(x_1, x_2) \in \mathcal{L}$ and all $(y_1, y_2) \in \mathbb{R}_+^2$, if $(y_1, y_2) \geq (x_1, x_2)$, then $(y_1, y_2) \in \mathcal{L}$. As $(X_t)_{t\geq 0}$ is non decreasing, this means that once failed, the system cannot leave $\mathcal{L}$ any more and remains failed (until it is repaired). For illustrative purpose, three different shapes are envisioned for $\mathcal{L}$. For the first two shapes, the system may be considered as composed of two different units and for $i = 1, 2$, the marginal process $\left(X_t^{(i)}\right)_{t\geq 0}$ stands for the deterioration level of the i-th unit. Setting $L_i > 0$ to be the failure threshold for the i-th unit, the corresponding univariate failure time is

$$\sigma_{L_i}^{(i)} = \inf\left\{t \geq 0 | X_t^{(i)} \geq L_i\right\}.$$

Two classical structures are then envisioned for the two-units system, which leads to the following first two cases:

Case 1 The two units are set up into series. The time-to-failure of the whole system then is:

$$\begin{aligned}\min\left(\sigma_{L_1}^{(1)}, \sigma_{L_2}^{(2)}\right) &= \inf\left\{t \geq 0 | X_t^{(1)} \geq L_1 \text{ or } X_t^{(2)} \geq L_2\right\} \\ &= \inf\{t \geq 0 | X_t \notin [0, L_1[\times[0, L_2[\} \\ &= \sigma_{\mathcal{L}},\end{aligned}$$

with $\mathcal{L} = \mathbb{R}_+^2 \backslash [0, L_1[\times[0, L_2[$.

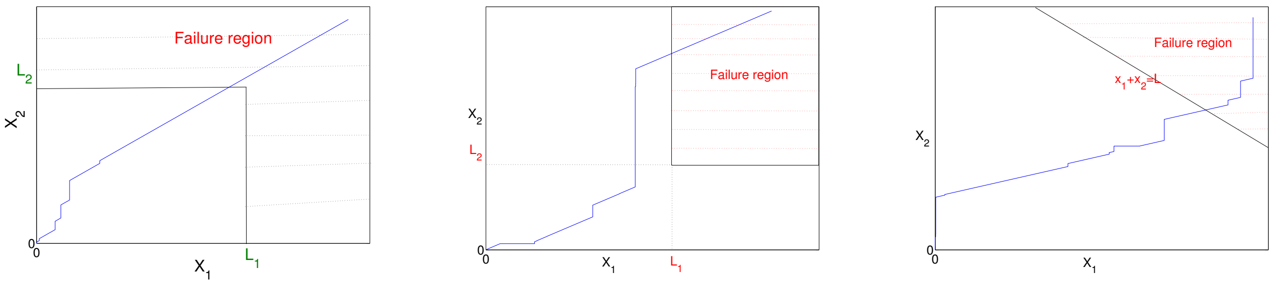
Case 2 The two units are set up into parallel. The time-to-failure of the whole system is $\max\left(\sigma_{L_1}^{(1)}, \sigma_{L_2}^{(2)}\right) = \sigma_{\mathcal{L}}$, with $\mathcal{L} = [L_1, \infty[\times[L_2, \infty[$.

Case 3 Both components of $(X_t)_{t\geq 0}$ stand for different indicators of a single system and the system time to failure is

$$\inf \left\{ t \geq 0 | X_t^{(1)} + X_t^{(2)} \geq L \right\} = \sigma_{\mathcal{L}},$$

with $\mathcal{L} = \left\{ (x_1, x_2) \in \mathbb{R}_+^2 | x_1 + x_2 \geq L \right\}$.

Such three shapes are plotted in Figure 1.



(a) The first shape (b) The second shape (c) The third shape

Figure 1: Examples of failure regions

Once the system is failed, a signal is sent to the repair team and an instantaneous repair takes place at time $\sigma_{\mathcal{L}} + \tau$, where τ is the deterministic time required by the repair team to arrive (the delay). The repair is perfect, which means that at repair, both of the system deterioration indicators are reset to zero.

2.2. The preventive maintenance policy

Without any PM policy, the system is down from $\sigma_{\mathcal{L}}$ up to $\sigma_{\mathcal{L}} + \tau$. To shorten this down-time (of length τ), the following PM policy is applied: setting $\mathcal{M}$ to be a closed and non empty upper set such that $\mathcal{L} \subset \mathcal{M} \subset \mathbb{R}_+^2$, a signal is preventively sent to the repair team at time $\sigma_{\mathcal{M}}(\leq \sigma_{\mathcal{L}})$. The system is then perfectly and instantaneously repaired at time $\sigma_{\mathcal{M}} + \tau$. If $\sigma_{\mathcal{L}} < \sigma_{\mathcal{M}} + \tau$, a failure occurs before the repair and the down-time duration is $\sigma_{\mathcal{M}} + \tau - \sigma_{\mathcal{L}}$. On the contrary, if $\sigma_{\mathcal{L}} \geq \sigma_{\mathcal{M}} + \tau$, the system is repaired before failure and there is no down-time up to the repair. In each case, the down time up to the repair hence is $(\sigma_{\mathcal{M}} + \tau - \sigma_{\mathcal{L}})^+ = \max(\sigma_{\mathcal{M}} + \tau - \sigma_{\mathcal{L}}, 0)$.

The future evolution of the system after repair is assumed to be independent from its past, and stochastically identical to its initial evolution. Setting $(Z_t)_{t\geq 0}$ to be the process describing the maintained system, $(Z_t)_{t\geq 0}$ appears as a regenerative process with cycles delimited by repairs (and $t = 0$) and generic cycle length $\sigma_{\mathcal{M}} + \tau$. This is illustrated in Figure 2 where the horizontal axis corresponds to the time and the vertical one to the deterioration level, drawn as a one-dimensional level for sake of clarity. Note that, as a bivariate subordinator (with null drift) is a pure jump process, the failure zone $\mathcal{L}$ has a non zero probability to be reached at the same time as the system enters $\mathcal{M}$. This can be seen in Figure 2, where $\sigma_{\mathcal{M}} = \sigma_{\mathcal{L}}$ in the first cycle ($\sigma_{\mathcal{M}}^{(1)} = \sigma_{\mathcal{L}}^{(1)}$). In the second cycle (which starts at $\sigma_{\mathcal{M}}^{(1)} + \tau$), the system is replaced before failure ($\sigma_{\mathcal{M}}^{(2)} + \tau < \sigma_{\mathcal{L}}^{(2)}$).

Taking $\mathcal{M} = \mathcal{L}$, the unmaintained system appears as a special case of the maintained system. Taking $\mathcal{M} = \mathbb{R}_+^2$ provides $\sigma_{\mathcal{M}} = 0$ and the system is periodically replaced, every τ time units. The classical periodic replacement policy with no repair at failure and period τ then appears as a special case of the PM policy.

To assess the PM policy, a cost function is considered, which takes into account:

- $C_1 > 0$: the restoration cost of the system,

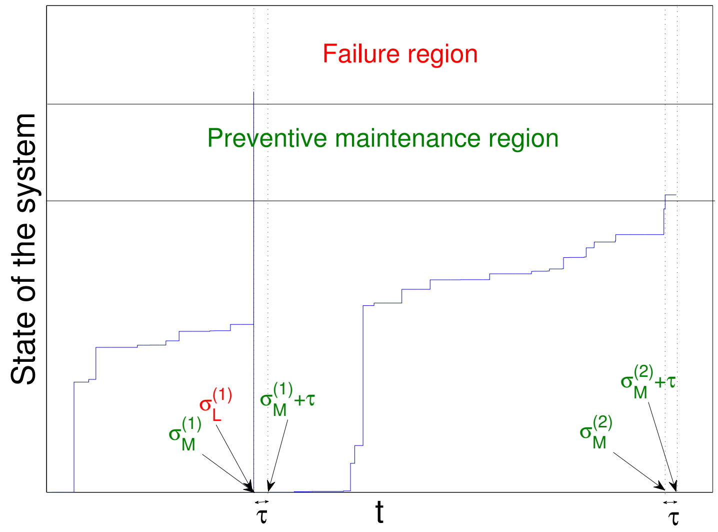


Figure 2: The preventive maintenance policy

- $C_2 > 0$: the unitary cost (per unit time) for down-time.

The envisioned cost function is the asymptotic unitary cost (per unit time), namely the function C_∞ defined by:

$$C_\infty = \lim_{t\to\infty} \frac{C(t)}{t} \text{ a.s.,}$$

where $C(t)$ stands for the accumulated cost on the time interval $[0, t]$. Our goal is to prove existence of C_∞, find a computable expression for it and study its behavior with respect to different parameters.

We will sometimes complete the assessment of the PM policy by another criterion, the asymptotic availability, defined by:

$$A_\infty = \lim_{t\to\infty} \frac{U(t)}{t} \text{ a.s.,}$$

where $U(t)$ stands for the system up-time on $[0, t]$. Methods are quite similar for both criteria and details are only provided for C_∞.

3. THE THEORETICAL RESULTS

3.1. Calculation of the cost function

Under the assumption $\mathcal{H}$, the means of $\sigma_{\mathcal{L}}$ and $\sigma_{\mathcal{M}}$ are finite. Thanks to the renewal theory (see in [2]), it implies that the asymptotic unitary cost exists a.s and it is equal to the unitary cost in each generic cycle:

$$C_\infty = \frac{C_1 + C_2 \mathbb{E}\left[(\sigma_{\mathcal{M}} + \tau - \sigma_{\mathcal{L}})^+\right]}{\tau + \mathbb{E}(\sigma_{\mathcal{M}})}. \tag{1}$$

By setting $\mathcal{L} - x = \{(y_1 - x_1, y_2 - x_2) | (y_1, y_2) \in \mathcal{L}\}$ for all $x = (x_1, x_2) \notin \mathcal{L}$ and

$$G_t(\mathcal{M}) = \mathbb{P}(X_t \in \mathcal{M}) = \iint_{\mathcal{M}} \mathbb{P}_{X_t}(dx_1, dx_2),$$
$$\overline{G}_t(\mathcal{M}) = \mathbb{P}(X_t \notin \mathcal{M}) = 1 - G_t(\mathcal{M}),$$

we obtain

$$h(\mathcal{M}) = \mathbb{E}(\sigma_{\mathcal{M}}) = \int_0^\infty \overline{G}_t(\mathcal{M})dt,$$

$$g_{\mathcal{L}}(\mathcal{M}) = \mathbb{E}\left[(\sigma_{\mathcal{L}} - \sigma_{\mathcal{M}} - \tau)^+\right] = \int_0^\infty \iint_{\mathcal{M}} \overline{G}_\tau(\mathcal{L} - x)\mathbb{P}_{X_v}(dx_1, dx_2)\ dv,$$

where $\mathbb{P}_{X_t}$ stands for the probability distribution of X_t . This provides the following expression for C_∞:

$$C_\infty = C_2 + \frac{C_1 + C_2(g_{\mathcal{L}}(\mathcal{M}) - h(\mathcal{L}))}{\tau + h(\mathcal{M})}, \tag{2}$$

The details of proof can be found in [7] as for the remaining of the paper.

Using similar methods as for C_∞, the asymptotic availability may be proved to exist almost surely and to be equal to the mean up time on a cycle divided by the mean cycle length:

$$A_\infty = \frac{\mathbb{E}(\sigma_{\mathcal{L}}) - \mathbb{E}\left[(\sigma_{\mathcal{L}} - \sigma_{\mathcal{M}} - \tau)^+\right]}{\tau + \mathbb{E}(\sigma_{\mathcal{M}})} = \frac{h(\mathcal{L}) - g_{\mathcal{L}}(\mathcal{M})}{\tau + h(\mathcal{M})},$$

which involves the same quantities as C_∞, and can consequently be computed at the same time.

As particular cases:

- For $\mathcal{M} = \mathcal{L}$ (unmaintained case), noting that $g_{\mathcal{L}}(\mathcal{L}) = 0$, we get:

$$C_\infty^{(ini)} = \frac{C_1 + C_2\tau}{\tau + \mathbb{E}(\sigma_{\mathcal{L}})} \text{ and } A_\infty^{(ini)} = \frac{\mathbb{E}(\sigma_{\mathcal{L}})}{\tau + \mathbb{E}(\sigma_{\mathcal{L}})},$$

- For $\mathcal{M} = \mathbb{R}_+^2$ (periodic replacements), we get:

$$C_\infty^{(PR)} = \frac{C_1 + C_2\mathbb{E}\left[(\tau - \sigma_{\mathcal{L}})^+\right]}{\tau}$$

  and

$$A_\infty^{(PR)} = \frac{\mathbb{E}(\sigma_{\mathcal{L}}) - \mathbb{E}\left[(\sigma_{\mathcal{L}} - \tau)^+\right]}{\tau} = \frac{\mathbb{E}[\min(\sigma_{\mathcal{L}}, \tau)]}{\tau}.$$

3.2. Some comparison results

If $\frac{C_1}{C_2} \geq \mathbb{E}[\min(\tau, \sigma_{\mathcal{L}})]$ then we can prove that $C_\infty \leq C_\infty^{(PR)}$ (whatever $\mathcal{M}$ is) and the PM policy is better than a simple periodic replacement policy. As a special case ($\mathcal{M} = \mathcal{L}$), this result shows that if $\frac{C_1}{C_2} \geq \mathbb{E}[\min(\tau, \sigma_{\mathcal{L}})]$, then we also prove that $C_\infty^{(ini)} \leq C_\infty^{(PR)}$. That means the periodical replacement policy always decreases the cost function when compared to the unmaintained case, whatever the period τ is.

If $\frac{C_1}{C_2} \geq \mathbb{E}(\sigma_{\mathcal{L}})$, then $C_\infty \geq C_\infty^{(ini)}$ and the best is not to use the PM policy, namely call for the repair team only at $\sigma_{\mathcal{L}}$. As a consequence from the previous results, if $\frac{C_1}{C_2} \geq \mathbb{E}(\sigma_{\mathcal{L}})$, then $C_\infty^{(ini)} \leq C_\infty \leq C_\infty^{(PR)}$. Also, the only situation where the PM policy can be interesting (namely s.t. $C_\infty \leq C_\infty^{(ini)}$) is the case where $\frac{C_1}{C_2} < \mathbb{E}(\sigma_{\mathcal{L}})$.

3.3. Influence of the delay time τ on C_∞

Though the delay time τ is generally fixed by the application context (and stands for the time required by the repair team to be ready to operate), we here consider that τ may vary, to better understand its influence both on the maintained and unmaintained system and write $C_\infty(\tau)$ instead of C_∞. We get the following results:

If $\mathbb{E}(\sigma_{\mathcal{L}}) < \frac{C_1}{C_2}$, the cost function $C_\infty(\tau)$ is decreasing with respect of τ, whatever $\mathcal{M}$ is. On the other hand, if $\mathbb{E}(\sigma_{\mathcal{L}}) \geq \frac{C_1}{C_2}$, assuming $\mathcal{L} \subsetneq \mathcal{M}$ and noting that $\mathbb{P}(\sigma_{\mathcal{L}} = \sigma_{\mathcal{M}})\,\mathbb{E}(\sigma_{\mathcal{M}}) < \mathbb{E}(\sigma_{\mathcal{L}})$, we have the following dichotomy:

- if $\frac{C_1}{C_2} \leq \mathbb{P}(\sigma_{\mathcal{L}} = \sigma_{\mathcal{M}})\,\mathbb{E}(\sigma_{\mathcal{M}})$ then the cost function $C_\infty(\tau)$ is non decreasing with respect of τ
- if $\mathbb{P}(\sigma_{\mathcal{L}} = \sigma_{\mathcal{M}})\,\mathbb{E}(\sigma_{\mathcal{M}}) < \frac{C_1}{C_2} \leq \mathbb{E}(\sigma_{\mathcal{L}})$ then the cost function $C_\infty(\tau)$ admits a unique minimum at some $\tau_{\mathcal{M}}$ $(0 < \tau_{\mathcal{M}} < +\infty)$ such that:

$$\int_0^{\tau_{\mathcal{M}}} \mathbb{P}(t < \sigma_{\mathcal{L}} - \sigma_{\mathcal{M}} \leq \tau_{\mathcal{M}})\,dt + \mathbb{P}(\sigma_{\mathcal{L}} - \sigma_{\mathcal{M}} \leq \tau_{\mathcal{M}})\,\mathbb{E}(\sigma_{\mathcal{M}}) - \frac{C_1}{C_2} = 0.$$

The behavior of the cost function with respect of τ may be quite different according to the case. As an example, in case of a high replacement cost $(\mathbb{E}(\sigma_L) < \frac{C_1}{C2})$, we can see that, from a cost point of view, the best is not to ever repair the system. Even if some benefit for up- time were considered in the cost function, such a result would still be valid in case of too high a replacement cost. In this situation, the system does not bring any profit , with or without preventive maintenance. If there is still some interest in the functioning of the system (which may be some client satisfaction e.g.), one should then control another reliability indicator, such as the system availability. It is easy to check that the system availability is always decreasing with τ. The optimal value of τ may then be provided by optimizing the cost function under some availability constraint, namely chose the largest τwhich meets with the availability constraint. As an alternative, one may also optimize the availability under some cost constraint, namely chose the shortest τ which meets with the cost constraint. More generally, from a cost point of view, one can observe that it is not necessarily mandatory that the repair team arrives as soon as possible (with the shortest τ) and some added delay in the repair may improve the cost function. However, such an added delay always decreases the availability, and it should then be controlled.

4. NUMERICAL EXPERIMENTS

4.1. A bivariate Gamma process

Let us first recall that an univariate Gamma process with parameters (a, b) (where $a, b > 0$) is a subordinator such that for every $t \geq 0$, the random variable Y_t is Gamma distributed $\Gamma(at, b)$ with probability distribution function (p.d.f.):

$$f_{at,b}(x) = \frac{1}{\Gamma(at)} b^{at} e^{-bx} x^{at-1} 1_{\{x>0\}}.$$

We only envision the case $b = 1$ in the following (no restriction) and we set $f_{at,b} = f_{at}$. The corresponding cumulative distribution function (c.d.f.) and survival function are denoted by F_{at} and $\bar{F}_{at}$, respectively, with $\bar{F}_{at} = 1 - F_{at}$.

Starting from three independent univariate Gamma processes $\left(Y_t^{(i)}\right)_{t\geq 0}$ with parameters $(\alpha_i, 1)$ for $i = 1, 2, 3$ (where $\alpha_1, \alpha_2, \alpha_3 > 0$), we set

$$X_t^{(1)} = Y_t^{(1)} + Y_t^{(3)} \text{ and } X_t^{(2)} = Y_t^{(2)} + Y_t^{(3)}.$$

The process $(X_t)_{t\geq 0} = \left(X_t^{(1)}, X_t^{(2)}\right)_{t\geq 0}$ then is a bivariate subordinator with Gamma marginal processes and marginal parameters $(a_i, 1)$ where $a_i = \alpha_i + \alpha_3$ for $i = 1, 2$. The linear correlation coefficient between the two random variables $X_t^{(1)}$ and $X_t^{(2)}$ is independent of t and given by

$$\rho = \frac{\alpha_3}{\sqrt{a_1 a_2}}.$$

We consequently have $\alpha_1 = a_1 - \rho\sqrt{a_1 a_2}$, $\alpha_2 = a_2 - \rho\sqrt{a_1 a_2}$ and $\alpha_3 = \rho\sqrt{a_1 a_2}$, with $0 \leq \rho \leq \rho_{\max} = \min\left(\sqrt{\frac{a_1}{a_2}}, \sqrt{\frac{a_2}{a_1}}\right)$, see [5] e.g.. Two equivalent alternate parameterizations hence are available for $(X_t)_{t\geq 0}$: either $(\alpha_1, \alpha_2, \alpha_3)$ or (a_1, a_2, ρ).

When $\rho \neq 0$, the joint p.d.f. of the random vector $X_t = (X_t^{(1)}, X_t^{(2)})$ is provided by:

$$f_{X_t}(x_1, x_2) = \int_0^{+\infty} f_{\alpha_1 t}(x_1 - x_3) f_{\alpha_2 t}(x_2 - x_3) f_{\alpha_3 t}(x_3) dx_3.$$

When $\rho = 0$, $X_t^{(1)}$ and $X_t^{(2)}$ are independent. Thus the joint p.d.f. of the random vector $X_t = (X_t^{(1)}, X_t^{(2)})$ is

$$f_{X_t}(x_1, x_2) = f_{a_1 t}(x_1) f_{a_2 t}(x_2).$$

4.2. Validation of the numerical results

Both C_∞ and A_∞ are here computed on a few examples, via the previous analytical results and by Monte-Carlo (MC) simulations, with 10^4 stories. For the MC results, the regenerative property of the system is exploited to derive some 95% confidence bands for the results, see[2] e.g.. We consider the three different cases for the shape of $(\mathcal{M}, \mathcal{L})$.

Case 1 We take $a_1 = 4$, $a_2 = 5$, $\rho = 0.6708$, $\tau = 0.1$, $M_1 = 3.4$, $M_2 = 2.4$, $L_1 = 3.5$, $L_2 = 2.5$, $C_1 = 100$ and $C_2 = 30$. The results are displayed in Table 1.

	Analytical formula	MC simulations	MC 95% confidence interval
C_∞	154.21612	154.38232	[152.79936; 155.96529]
A_∞	0.87203	0.87245	[0.85885; 0.88604]

Table 1: Comparison with MC simulations, Case 1 (series system)

Case 2 We take $a_1 = 7$, $a_2 = 9$, $\rho = 0.75$, $\tau = 0.1$, $M_1 = 2.9$, $M_2 = 2.3$, $L_1 = 3.5$, $L_2 = 2.5$, $C_1 = 100$ and $C_2 = 30$. The results are displayed in Table 2.

	Analytical formula	MC simulations	MC 95% confidence interval
C_∞	172.60371	171.04858	[168.722395; 173.3732]
A_∞	0.91734	0.917911	[0.90132; 0.93450]

Table 2: Comparison with MC simulations, Case 2 (parallel system)

Case 3 We take $a_1 = 4$, $a_2 = 9$, $\rho = 0.4$, $\tau = 0.1$, $M = 2.4$, $L = 3.5$, $C_1 = 3$ and $C_2 = 1$. The results are displayed in Table 3.

Such comparison results validate the theoretical results from the previous section.

	Analytical formula	MC simulations	MC 95% confidence interval
C_∞	9.0611	9.0461	[8.6960; 9.3961]
A_∞	0.8750	0.8738	[0.8407; 0.9069]

Table 3: Comparison with MC simulations, Case 3

4.3. Examples

We now illustrate our results through different numerical experimentations. The parameters of the bivariate Gamma process and the shape of $(\mathcal{M}, \mathcal{L})$ are provided in Table 4 for each example. Here the cases 1, 2 and 3 refer to the different cases from the subsection 2.1.

	a_1	a_2	ρ	τ	shape of $(\mathcal{M}, \mathcal{L})$	L_1 (ou L)	L_2	M_1 (ou M)	M_2	C_1	C_2
Ex. 1	4	9	0.5	-	case 1	3.5	2.5	2.8	2	–	1
Ex. 2	7	9	0.76	0.1	case 2	3.5	2.5	-	-	-	1
Ex. 3	4	9	-	0.1	case 1	3.5	2.5	2.8	2	20	-
	7	9	-	0.1	case 2	3.5	2.5	2.9	2.3	20	-
Ex. 4	4	-	-	0.1	case 3	3.5		2.4		1	-

Table 4: Parameters and shapes of $(\mathcal{M}, \mathcal{L})$ for the different examples

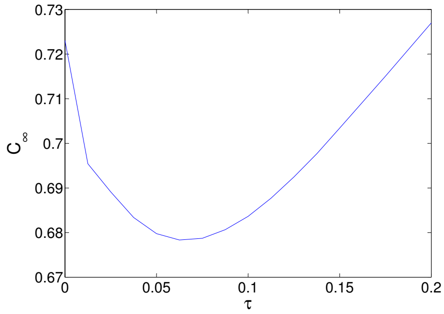
Example 1 *Two different values are considered for C_1 : $C_1 = 0.198$ and $C_1 = 0.594$, and C_∞ is plotted against the delay time τ in Figure 3 for both values. In the first case (Figure 3a), C_1 is such that $\frac{C_1}{C_2} < \mathbb{E}(\sigma_{\mathcal{L}})$ and the cost function C_∞ is minimum at $\tau_0^{opt} \simeq 0.0625$. In the second case (Figure 3b), we have $\frac{C_1}{C_2} > \mathbb{E}(\sigma_{\mathcal{L}})$ and C_∞ is decreasing with τ. In case $\frac{C_1}{C_2} > \mathbb{E}(\sigma_{\mathcal{L}})$, the lowest cost is obtained for $\tau = \infty$ which means that, from a cost point of view, the best is not to ever repair the system. Thus one should then control the asymptotic availability, which is also decreasing in τ, see Figure 3c. Assume for instance that we have an availability constraint provided by $A_\infty \geq 0.9$ (to ensure client satisfaction e.g.). The optimal value of τ which minimizes the cost function under this availability constraint then is the largest τ which fullls this constraint, namely $\tau_0 \simeq 0.075$.*

Example 2 *Two different values are considered for C_1 : $C_1 = 0.15$ and $C_1 = 2$, and C_∞ is plotted against (M_1, M_2) in Figure 4 for both values. In the first case (Figure 4a), C_1 is such that $\frac{C_1}{C_2} < \mathbb{E}(\sigma_{\mathcal{L}})$ and the cost is minimum at $(M_1^{opt}, M_2^{opt}) \simeq (2.8, 1.8)$. In the second case, we have $C_1 > C_2\mathbb{E}(\sigma_{\mathcal{L}})$ (Figure 4b) and the cost is minimum at (L_1, L_2), which means that no PM policy is required.*

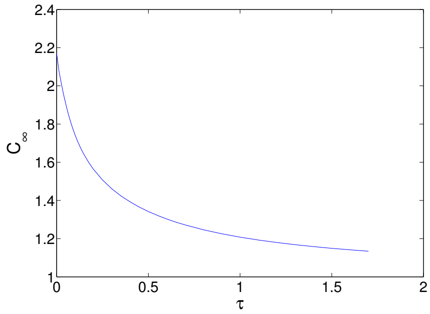
Example 3 *We take three different values for C_2: $C_2 = 4$, $C_2 = 20$ and $C_2 = 30$, and two different shapes for $(\mathcal{M}, \mathcal{L})$. The cost is plotted against the dependence (measured by ρ) in Figure 5 in all these cases. For the first shape of $(\mathcal{M}, \mathcal{L})$, we observe that the cost is decreasing with ρ for the three values of C_2 (Figure 5a). For the second shape of $(\mathcal{M}, \mathcal{L})$, the monotony is reversed (Figure 5b) and the cost is increasing with ρ.*

Example 4 *We here consider the third shape for $(\mathcal{M}, \mathcal{L})$ and the cost is plotted against ρ for four different couples (a_2, C_2) in Figure 6, with $(a_2, C_2) \in \{(9,1), (9,10), (4,30), (4,7)\}$.*

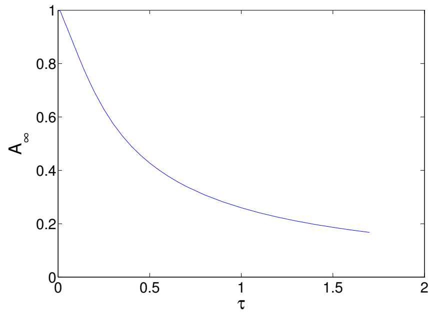
According to these four cases, we can see that the cost may be increasing, decreasing, concave, convex with respect of ρ, so that nothing can be said about the behavior of the cost function with respect of the dependence.



(a) C_∞, case $\frac{C_1}{C_2} < \mathbb{E}(\sigma_L)$

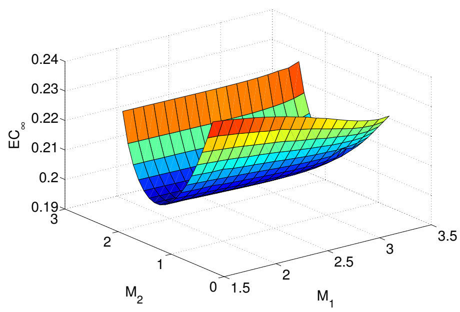


(b) C_∞, case $\frac{C_1}{C_2} > \mathbb{E}(\sigma_L)$

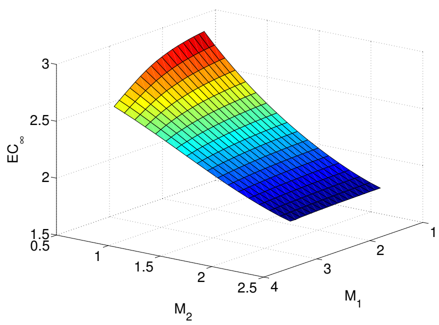


(c) A_∞, case $\frac{C_1}{C_2} > \mathbb{E}(\sigma_L)$

Figure 3: C_∞ and A_∞ as a function of τ, Example 1.



(a) Case $C_1/C_2 < \mathbb{E}(\sigma_L)$



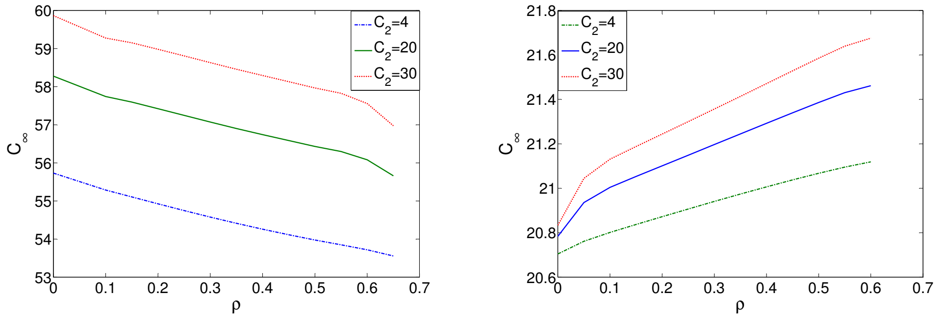
(b) Case $C_1/C_2 > \mathbb{E}(\sigma_L)$

Figure 4: C_∞ as a function of (M_1, M_2), Example 2.

5. CONCLUSION

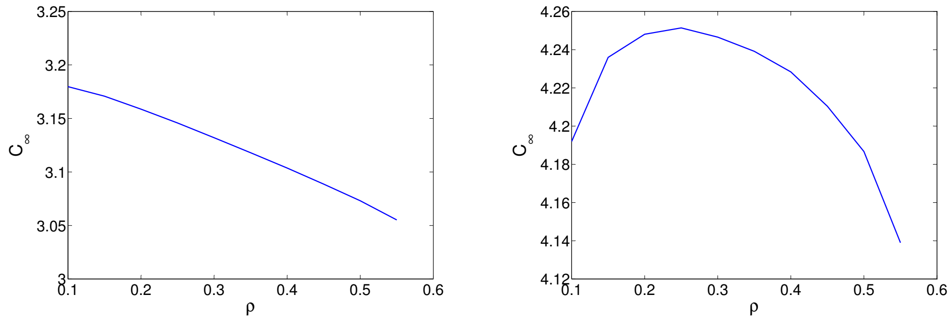
We here proposed a PM policy for a continuously monitored system modeled by a bivariate subordinator. The PM policy has been assessed through a cost function on an infinite horizon time. We studied some conditions under which the PM policy decreases the cost function when compared to a simple periodic replacement policy or to the unmaintained case. The influence of the delay time on the cost function has been studied too.

As for the influence of the dependence between the two wear indicators on the cost function, we have not been able to study it from a theoretical point of view. We have however numerically observed that the cost function seemed to be monotonic with respect to the dependence for the first two shapes of failure regions considered in the paper (decreasing for the first shape and increasing for the second one). The proof of these monotonicity results remains a challenging open question. As for the third envisioned shape, the different examples show that the cost function is not monotonic with the

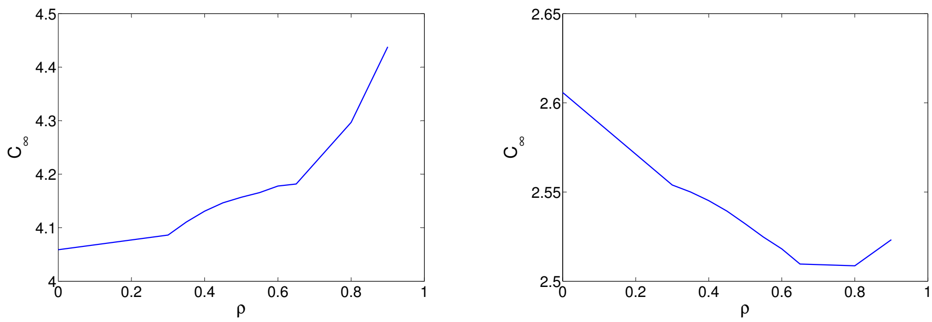


(a) Case 1 - the first shape (b) Case 2 - the second shape

Figure 5: C_∞ as a function of ρ, Example 3.



(a) $a_2 = 9, C_2 = 1$ (b) $a_2 = 9, C_2 = 10$



(c) $a_2 = 4, C_2 = 30$ (d) $a_2 = 4, C_2 = 7$

Figure 6: C_∞ as a function of ρ, Example 4, Case 3 - the third shape.

dependence. The shape of the failure region hence has a clear influence on the eventual monotonicity of the cost function with respect to the dependence. According to the case, not taking into account the dependence between the wear indicators may hence lead to under- or over-estimate the cost function (see Figures 5 and 6), which may induce difficulties in an industrial context. Taking into account the dependence as in the present paper then is essential.

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