Asymptotic failure rate of a Markov deteriorating system with preventive maintenance

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Abstract

We consider a system with a finite state space subject to continuous-time Markovian deterioration while running, that leads to failure. Failures are instantaneously detected. This system is submitted to sequential checking and preventive maintenance: up-states are divided into "good" and "degraded" ones and the system is sequentially checked through perfect and instantaneaous inspections until it is found in a degraded up-state and stopped to be maintained (or until it fails). Time between inspections is random and is chosen at each inspection according to the observed degradation degree of the system. Maintenance duration follows general distribution. Markov renewal equations fulfilled by the reliability of the maintained system are given and an exponential equivalent is derived for the reliability. We prove the existence of an asymptotic failure rate for the maintained system, which we are able to compute. We show that under natural conditions on the preventive maintenance policy, the reliability and the asymptotic failure rate of the system are both improved by the preventive maintenance policy.

Key-words: Preventive maintenance; Sequential checking; Markov renewal equation; Asymptotic failure rate.

1 Introduction

We consider a system with a finite state space subject to continuous-time Markovian deterioration while running, that leads to failure: typically, we are here concerned with a system formed of components with constant failure rates, which cannot be repaired while the system is running. However, they may be repaired (or replaced) during a stop of the system for preventive maintenance, with general repair rates. No continuous monitoring is performed on the system, so that the state of the running system is not continuously known. However, failures are instantaneously detected. Also, it is possible to know the current state of the running system (among a finite number of possible up-states) through perfect instantaneous inspections, which do not degrade the system. The system is then submitted to the following preventive maintenance policy: the up-states are divided into two parts, "good" ones and "degraded" ones, and the system is inspected until it is found in a degraded up-state and stopped to be maintained for a random duration that depends on the degradation degree of the system. Time between inspections may be random and is chosen at each inspection according to the observed state of the system. Such a dynamic succession of inspections had been called *sequential* checking procedure by Barlow and al. (1963) (to be opposed to *periodic* checking procedure).

The same modelization has already been studied in Cocozza-Thivent (2000) and Bloch-Mercier (2002), where the authors are respectively interested in the availability and in the mean cost per unit time in long-time run. Though, we are here only concerned with the system up to its first failure (with possible preventive maintenance actions however) whereas such papers are concerned with the system in long-time run with possible *corrective* maintenance actions. Note that, in an industrial context, it may be interesting to control different criteria, though optimization of the preventive maintenance policy is generally lead with respect to one single criterion.

The proofs of the different results, a numerical example and some more references may be found in Bloch-Mercier and Roussignol (2001).

2 Notations - Assumptions

Let 1, 2, ..., m be the up-states of the system and let $m+1$, ..., $m+p$ be the down-states. The system is up at the beginning and then evolves in time according to the Markov process $(X_t)_{t\geq 0}$ as long as it is running. (We consider the down-states as absorbing for (X_t)). The system almost surely breaks down after a finite time T^{ini}: $\mathrm{P}_i\left(T^{ini} < +\infty\right) = 1$, where P_i is the conditional probability, given that the system started from state i (all $1 \leq i \leq m$). Symbol E_i represents the associated conditional expectation. Symbol $(P_t(i,j))$ represents the transition semi-group associated to (X_t) : $P_t(i,j) = \mathrm{P}_i(X_t = j)$ (all $1 \leq i,j \leq m+p$), with $P_t(i,j) = 0$, all $m+1 \leq i \leq m+p$, $1 \leq j \leq m+p$.

Given that the system started in state i ($1 \leq i \leq m$), we recall that its reliability is:

$$R_i^{ini}(t) = \mathrm{P}_i\left(T^{ini} > t\right) = \sum_{j=1}^{m} \mathrm{P}_i(X_t = j) = \sum_{j=1}^{m} P_t(i,j) \qquad \text{(all } t \geq 0\text{)}$$

Exponent *ini* here refers to the *initial* or unmaintained system, to be opposed to the *maintained* system, submitted to the following preventive maintenance policy:

The up-states are divided between "good" up-states: $1, ..., q$ and "degraded" ones: $q+1, ..., m$, with q a fixed integer such that $1 \leq q \leq m-1$.

Let ρ_1, ρ_2, ..., ρ_m be some probability distributions on $\mathbb{R}_+$ such that $\int_0^{+\infty} t.\rho_i(dt) > 0$, all $1 \leq i \leq m$.

Now, let $S_0 = 0$ and let S_1 be a random variable independent of the evolution of the system, with ρ_{X_0} as distribution (we recall that $X_0 \in \{1, ..., m\}$). As long as the system is not failed, it is instantaneously inspected by time S_1, S_2, ..., S_n, ... recursively defined in the following way (all $n \in \mathbb{N}^*$):

- If $X_{S_n} = j \in \{1, ..., q\}$, the system is in a "good" up-state. We do nothing but choosing the time to next inspection: next inspection will take place at time $S_{n+1} = S_n + U_n$, where U_n is a random variable with ρ_j for distribution, independent of the previous evolution of the system before S_n. (The random variable U_n only depends on the state $X_{S_n} = j$ of the system at time of inspection S_n).

- If $X_{S_n} = j \in \{q+1, ..., m\}$, the system is in a "degraded" up-state. The system is stopped and a maintenance action is instantaneously begun. This maintenance action leaves the system in a state which is assumed to be independent of the previous evolution of the system before S_n and the system starts again in the up-state k ($1 \leq k \leq m$) with probability $D(j,k)$. Given that the system starts again in state k, the duration of the maintenance action has the same distribution $\mu_{j,k}$ as a random variable $M_{j,k}$, independent of the previous evolution of the system. The system cannot fail during a maintenance action.

After a preventive maintenance action, a new sequence of inspections is begun, recursively defined in the same way as from the beginning.

The time to failure of the maintained system is represented by T so that the reliability of the maintained system leaving from state i (all $1 \leq i \leq m$) is $R_i(t) = \mathrm{P}_i(T > t)$.

3 Markov renewal equations

Let $*$ represent the convolution between two measures and let $P_.(i,j)\,\rho_i$ be the measure with density $P_s(i,j)$ with respect to $\rho_i(ds)$ (namely: $(P_.(i,j)\,\rho_i)(ds) = P_s(i,j)\,\rho_i(ds)$). Using some renewal property according to what happens by time S_1 of the first inspection (whenever it takes place), we get the following result:

Theorem 1 *The reliability of the maintained system $R_i(t)$ leaving from state i fulfills the following Markov renewal equations:*

$$R_i(t) = G_i(t) + \sum_{j=1}^{m} \int_{[0,t]} R_j(t-s)\,\nu_{i,j}(ds) \qquad (\text{all } 1 \le i \le m) \tag{1}$$

with

$$\nu_{i,j}(ds) = \mathbf{1}_{\{1\le j\le q\}} P_s(i,j)\,\rho_i(ds) + \sum_{k=q+1}^{m} D(k,j)\left((P_\cdot(i,k)\,\rho_i) * \mu_{k,j}\right)(ds) \tag{2}$$

$$G_i(t) = R_i^{ini}(t)\,\mathrm{P}_i(S_1 > t) + \sum_{k=q+1}^{m}\sum_{j=1}^{m}\int_{[0,t]} \rho_i(ds)\,\mu_{k,j}(]t-s,+\infty[)\,P_s(i,k)\,D(k,j) \tag{3}$$

4 Asymptotic behaviour of the reliability

From those Markov renewal equations, we now derive some indications on the asymptotic behaviour of the reliability of the maintained system. We first note that symbol $\nu_{i,j}(ds)$ that appears in (1) is not a semi-Markov kernel in the sense that $\sum_{j=1}^{m}\int_0^{+\infty}\nu_{i,j}(ds) < 1$. The results of this Section are then based on Asmussen (1987), Chapter X, Thm 2.6: the idea is to transform equations (1) to boil down to new equations associated to a true semi-Markov kernel, and then apply the key renewal theorem (or some generalization). In that aim, we define, following Asmussen, the matrix $\bar{A}^\beta = \left(a_{i,j}^\beta\right)_{1\le i,j\le m}$ such that

$$a_{i,j}^\beta = \int_0^{+\infty} e^{\beta u}\nu_{i,j}(du) = \mathbf{1}_{\{1\le j\le q\}}\mathrm{E}_i\left(e^{\beta S_1}P_{S_1}(i,j)\right) + \sum_{k=q+1}^{m}\mathrm{E}_i\left(e^{\beta S_1}P_{S_1}(i,k)\right)\mathrm{E}\left(e^{\beta M_{k,j}}\right)D(k,j)$$

and we get:

Theorem 2 *Let us assume that:*

$$\begin{cases} (H_1) \text{ Matrix } \bar{A}^0 \text{ has got an irreducible component } I, \\ (H_2) \text{ The Laplace transform of } \rho_i \text{ is convergent on } \mathbb{R} \text{ (all } i\in I), \\ (H_3) \text{ The Laplace transform of } \mu_{k,j} \text{ is convergent on } \mathbb{R} \\ \text{(all } q+1\le k\le m,\ j\in I \text{ such that } D(k,j)>0), \\ (H_4) \text{ There is some } j_0 \text{ in } I \text{ and some } k_0\ (q+1\le k_0\le m) \\ \text{such that } D(k_0,j_0)>0 \text{ and such that measure } \mu_{k_0,j_0} \\ \text{admits a density towards Lebesgue measure.} \end{cases}$$

Then, there is some β_0 such that A^{β_0} has spectral radius 1 ($\beta_0 > 0$), where $A^{\beta_0} = \left(a_{i,j}^{\beta_0}\right)_{(i,j)\in I^2}$. For such a β_0, let $\boldsymbol{\eta}^{\beta_0}$ and $\boldsymbol{\xi}^{\beta_0}$ positive componentwise such that $A^{\beta_0}\boldsymbol{\eta}^{\beta_0} = \boldsymbol{\eta}^{\beta_0}$ and $\boldsymbol{\xi}^{\beta_0}A^{\beta_0} = \boldsymbol{\xi}^{\beta_0}$. Then:

$$\lim_{t\to+\infty} e^{\beta_0 t}R_i(t) = \boldsymbol{\eta}_i^{\beta_0}\frac{\sum_{j\in I}\boldsymbol{\xi}_j^{\beta_0}\int_0^\infty e^{\beta_0 u}G_j(u)\,du}{\sum_{i\in I}\sum_{j\in I}\boldsymbol{\xi}_i^{\beta_0}\boldsymbol{\eta}_j^{\beta_0}\int_0^\infty u.e^{\beta_0 u}\nu_{i,j}(du)} =_{(say)} L_i\ ,\ \text{all}\ \ i\in I$$

with

$$0 < L_i < +\infty$$

As irreducibility of $\bar{A}^0$ is required for the key renewal theorem, we are lead to assume the existence of *an irreducible component* I for $\bar{A}^0$ (H_1) and to apply the results of Asmussen (1987) on the set I. Assumptions (H_2) and (H_3) ensures us with the existence of β_0 such that A^{β_0} has spectral radius 1. To apply the key renewal theorem, we actually use alternative conditions to those used by Asmussen and we show that, under (H_{2-3}), the function $e^{\beta_0 t}G_i^{\beta_0}(t)$ is bounded, Lebesgue-integrable and tends to zero when t goes to infinity. Under (H_{2-4}), it may also be shown that the condition of aperiodicity is checked too.

Under assumptions (H_{1-3}), we will now refer to symbol β_0 in the following without any more precision.

5 Asymptotic failure rate

Under (H_{1-4}) and for $i \in I$, we write the asymptotic failure rate in the following way:

$$\lim_{t\to+\infty}\lim_{h\to0^+}\frac{1}{h}\mathrm{P}_i\left(T\le t+h|T>t\right)=-\frac{\lim_{t\to+\infty}e^{\beta_0 t}R_i'^{+}\left(t\right)}{\lim_{t\to+\infty}e^{\beta_0 t}R_i\left(t\right)}$$

whenever the right-hand derivative $R_i'^{+}(t)$ of $R_i(.)$ and limit in numerator exist.

In order to prove the existence and to compute this expression, we first show the existence of $R_i'^{+}(t)$ and get Markov renewal equations for $R_i'^{+}(t)$. We then derive $\lim_{t\to+\infty}e^{\beta_0 t}R_i'^{+}(t)$ in the same way as for $R_i(t)$ (with the *same* β_0). We get:

Theorem 3 *Under assumptions (H_{1-4}): if the system starts from a state i in I, β_0 is the asymptotic failure rate for the maintained system, namely:*

$$\beta_0=\lim_{t\to+\infty}\lim_{h\to0^+}\frac{1}{h}P_i\left(T\le t+h|T>t\right)\qquad \text{all } i\in I$$

6 Comparison between the initial and the maintained system

We show here that, as expected, if the system is in a "better" state after a maintenance action than before (see (5) just below), then, the preventive maintenance policy improves the reliability of the system (any time). Symbol $\prec_{sto}$ stands for the usual stochastic ordering. Symbol $D(k,.)$ is the k^{th} row of matrix D and represents the distribution of the random state in which the system starts again after a maintenance action that began in state k, whereas symbol δ_k is the Dirac measure on $\{1,...,m\}$ at state k.

Theorem 4 *Under assumptions (H_{1-4}) and with the same notations, if, addingly:*

$$R_i^{ini}(t)\le\sum_{j=1}^{m}D(i,j)\,R_j^{ini}(t)\,,\ \text{all } q+1\le i\le m \tag{5}$$

or if both

$$R_i^{ini}(t)\ge R_{i+1}^{ini}(t)\,,\ \text{all } t\ge0,\ 1\le i\le m-1 \tag{6}$$

and

$$D(k,.)\prec_{sto}\delta_k,\ \text{all } q+1\le k\le m \tag{7}$$

are true, then:

$$R_i^{ini}(t)\le R_i(t)\,,\ \text{all } i\in I, t\ge0 \tag{8}$$

Besides, if β^{ini} exists such that $\lim_{t\to+\infty}e^{\beta^{ini}t}R_i^{ini}(t)$ is finite and positive (all $i\in I$), then: $\beta_0\le\beta^{ini}$

If the states are ranked according to their increasing degradation degree according to (6), assumption (7) then is an alternative to (5), somewhat easier to check.

References

Asmussen, S. (1987). *Applied Probability and Queues.* John Wiley & Sons, Chichester.

Barlow, R. E., Hunter, L. C., Proschan, F. (1963). Optimal checking procedures. *J. Soc. Indust. Appl. Math. 11, n°4,* 1078–1095.

Bloch-Mercier, S., Roussignol, M. (2001). Asymptotic failure rate of a Markov deteriorating system with preventive maintenance. *Prépub. de l'Université de Marne-la-Vallée n°12/2001*, submitted.

Bloch-Mercier, S. (2002). A Preventive maintenance policy with sequential checking procedure for a Markov deteriorating system, to appear in *European Journal of Operational Research.*

Cocozza-Thivent, C. (2000). A model for a dynamic preventive maintenance policy. *Journal of Applied Mathematics and Stochastic Analysis 13:4*, 321–346.