A Finite Volume Scheme for Sensitivity Analysis in Dynamic Reliability

Robert Eymard, Sophie Mercier, Alain Prignet, Michel Roussignol

Université Paris-Est,
Laboratoire d'Analyse et de Mathématiques Appliquées, UMR CNRS 8050,
5 boulevard Descartes, 77454 Marne la Vallée Cedex 2, France

{robert.eymard,sophie.mercier,alain.prignet,michel.roussignol}@univ-mlv.fr

ABSTRACT. *In dynamic reliability, the evolution of a system is modelled by a piecewise deterministic Markov process, which depends on different data. Assuming such data to depend on some real parameter* p*, our aim is to compute the derivative with respect to* p *of the mathematical expectation of some functions, in view of sensitivity analysis. Thanks to the use of an implicit finite volume scheme for approximating the marginal distributions of the piecewise deterministic Markov process, these derivatives can be calculated using the adjoint-state method. The efficiency of the method is proven by a numerical example, where the derivatives are computed both by finite differences and by the adjoint state method.*

KEYWORDS: *piecewise deterministic Markov process, importance factor*

1. Introduction

In dynamic reliability, the time-evolution of a system is described by a piecewise deterministic Markov process (PDMP) $(I_t, X_t)_{t\geq 0}$ (see [DAV 84]). The first component I_t is discrete, with values in a finite state space E. Typically, it indicates the state (up/down) for each component of the system at time t. The second component X_t, with values in a Borel set $V \subset \mathbb{R}^d$, stands for environmental conditions, such as temperature, pressure, and so on. Both components of the process interact with each other: the process jumps at many countably isolated random times; by a jump from $(I_{t^-}, X_{t^-}) = (i, x)$ to $(I_t, X_t) = (j, y)$ (with (i, x), $(j, y) \in E \times V$), the transition rate between the discrete states i and j depends on the environmental condition x just before the jump and is a function $x \longmapsto a(i, j, x)$. Similarly, the environmental condition just after the jump X_t is distributed according to a distribution $\mu_{(i,j,x)}(\mathrm{d}y)$, which depends on both components just before the jump (i, x) and on the post-jump discrete state j. For the sake of simplicity, we here assume $\mu_{(i,j,x)}(\mathrm{d}y)$ to be a Dirac measure: $\mu_{(i,j,x)}(\mathrm{d}y) = \delta_{F(i,j,x)}(\mathrm{d}y)$. Between jumps, the discrete component I_t

is constant, whereas the evolution of the environmental condition $X_t \in V \subset \mathbb{R}^d$ is deterministic, solution of a set of differential equations which depends on the fixed discrete state: given that $I_t(\omega) = i$ for all $t \in [a, b]$, we have $\frac{d}{\mathrm{d}t} X_t(\omega) = \mathbf{v}(i, X_t(\omega))$ for all $t \in [a, b]$, where $\mathbf{v}$ is a mapping from $E \times V$ to V. The jump rates $a(i, j, x)$, the jump function $F(i, j, x)$, the velocity fields $\mathbf{v}\,(i, x)$ are assumed to depend on some family of parameters $P \in \mathbb{R}^k$, where $k \in \mathbb{N}$ can be quite large.
Given such a PDMP $(I_t, X_t)_{t\geq 0}$, our aim is to provide information about the sensitivity with respect to the elements of P, of expressions given under the form of the mathematical expectation of some bounded measurable functions h (which can also depend on $p \in P$) of the process:

$$R_{\rho_0}(t) = \mathbb{E}_{\rho_0}\left(\int_0^t h(I_s, X_s)\ \mathrm{d}s\right)$$

where ρ_0 is the initial distribution of the process. Such expressions include e.g. cumulative availability or production availability on some $[0, t]$, mean number of failures on $[0, t]$, mean time spent by $(X_s)_{0\leq s\leq t}$ between two given bounds.
This sensitivity analysis can be guided by the knowledge of the first-order logarithmic derivatives of $R_{\rho_0}(t)$ with respect to p, where $p \in P$. Indeed, one can order the components of P, following the dimensionless expression

$$IF_p = \frac{p}{R_{\rho_0}(t)} \frac{\partial R_{\rho_0}(t)}{\partial p}, \qquad [1]$$

which we call the importance factor in $R_{\rho_0}(t)$ of the parameter $p \in P$. Note that such an expression only makes sense when considering a never vanishing parameter p, which we consequently assume to be positive. This kind of sensitivity analysis has already been studied by Gandini [GAN 90] for pure jump Markov processes with countable state space, and extended to PDMP in [MER 07], with a more restrictive model than in the present paper, however.
Since the marginal distributions of the process $(I_t, X_t)_{t\geq 0}$ are, in some sense, the weak solution of linear first order hyperbolic equations [3] [COC 06], the expressions for the derivatives of the mathematical expectations can be obtained by resolving the dual problem, as suggested in [LIO 68] for a wide class of partial differential equations. We show here that the resolution of the dual problem provides an efficient numerical method, when the marginal distributions of the PDMP are approximated using a finite volume method.

2. Theoretical results

For $p \in P$, exponent $^{(p)}$ is generally added to quantities depending on p, but is sometimes omitted, in order to prevent too cumbersome expressions. All the paper is written under the following assumptions $(\mathcal{H})$: let O be an open subset of $\mathbb{R}^+$; for each p in O, there is some neighbourhood $N(p)$ of p in O such that, for all $i, j \in E \times E$, the function $(x, p) \longmapsto a^{(p)}(i, j, x)$ is bounded on $V \times N(p)$, continuously differentiable on $V \times O$, with all partial derivatives uniformly bounded on $V \times N(p)$; the function $(x,p) \longmapsto F^{(p)}(i,j,x)$ is continuously differentiable on $V \times O$, bounded on $V \times N(p)$, with all partial derivatives uniformly bounded on $V \times N(p)$; for all $i \in E$, the function $(x,p) \longmapsto \mathbf{v}^{(p)}(i,x)$ is bounded on $V \times N(p)$ by some $C_{\mathbf{v},N(p)} = \max_{i\in E} \sup_{(x,q)\in V\times N(p)} \left|\mathbf{v}^{(q)}(i,x)\right| > 0$, continuously differentiable on $V \times O$, with all partial derivatives uniformly bounded on $V \times N(p)$; for all $i \in E$, the function $(x,p) \longmapsto h^{(p)}(i,x)$ is bounded on $V \times N(p)$, continuously differentiable on $V \times O$ with uniformly bounded partial derivatives on $V \times N(p)$.

We denote by $\rho_t^{(p)}(i, \mathrm{d}x)$ the distribution of the Markov process $\left(I_t^{(p)}, X_t^{(p)}\right)_{t\geq 0}$ at time t with initial distribution ρ_0 (independent on p). We then have

$$R_{\rho_0}^{(p)}(t) = \int_0^t \sum_{i\in E} \int_V h^{(p)}(i,x)\ \rho_s^{(p)}(i,\mathrm{d}x)\ \mathrm{d}s. \qquad [2]$$

In order to express $\frac{\partial}{\partial p} R_{\rho_0}^{(p)}(t)$, we first introduce the infinitesimal generator $H^{(p)}$ of the Markov process $(I_s, (X_s, s))_{s\geq 0}$:

Definition 1 *Let $\mathcal{D}_H$ be the set of functions $\varphi(i,x,s)$ from $E \times V \times \mathbb{R}_+$ to $\mathbb{R}$, such that for all $i \in E$ the function $(x,s) \longmapsto \varphi(i,x,s)$ is bounded, continuously differentiable on $V \times \mathbb{R}_+$ and such that the function $x \longmapsto \frac{\partial \varphi}{\partial s}(i,x,s) + \mathbf{v}^{(p)}(i,x) \cdot \nabla\varphi(i,x,s)$ is bounded on $V \times \mathbb{R}_+$. For $\varphi \in \mathcal{D}_H$, we define*

$$H^{(p)}\varphi(i,x,s) = \sum_{j\in E} a^{(p)}(i,j,x)\varphi(j, F^{(p)}(i,j,x), s) + \frac{\partial \varphi}{\partial s}(i,x,s) + \mathbf{v}^{(p)}(i,x)\cdot\nabla\varphi(i,x,s)$$

where $F^{(p)}(i,i,x) = x$ and $a^{(p)}(i,i,x) = -\sum_{j\neq i} a^{(p)}(i,j,x)$ for all $(i,x) \in E \times V$.

It is then known from [COC 06] that $\left(\rho_s^{(p)}(i,\mathrm{d}x)\right)_{s\geq 0}$ is the unique family of measures solution to:

$$\rho_s^{(p)}\varphi(\cdot,\cdot,s) = \rho_0^{(p)}\varphi(\cdot,\cdot,0) + \int_0^s \rho_u^{(p)} H^{(p)}\varphi(\cdot,\cdot,u)\ \mathrm{d}u \qquad [3]$$

for all $\varphi \in \mathcal{D}_H$ and all $s \in \mathbb{R}_+$ (Chapman-Kolmogorov equation).

For example, in case $F^{(p)}(i,j,x) = x$ and assuming regular enough data, the solution is expected to have a density with respect to Lebesgue measure: $\rho_t^{(p)}(i,\mathrm{d}x) = r^{(p)}(i,x,t)\mathrm{d}x$. The Chapman-Kolmogorov equation then is a weak form of

$$\frac{\partial r^{(p)}}{\partial t}(i,x,t) + \mathrm{div}\left(r^{(p)}(i,x,t)\mathbf{v}^{(p)}(i,x)\right) = \sum_{j\in E} a^{(p)}(j,i,x) r^{(p)}(j,x,t).$$

Next we introduce some functions, called importance functions:

Proposition 2 *Let $t > 0$. Under assumptions $\mathcal{H}$, there exists one and only one function $\varphi_t^{(p)} \in \mathcal{D}_H$ such that $\varphi_t^{(p)}$ is solution of the differential equation $H^{(p)}\varphi_t^{(p)}(i,x,s) = h^{(p)}(i,x)$ for all $(i,x,s) \in E \times V \times [0,t[$, with the initial condition $\varphi_t^{(p)}(i,x,t) = 0$ for all $(i,x) \in E \times V$. The function $\varphi_t^{(p)}$ is called the importance function associated with $\left(h^{(p)}, t\right)$.*

The following theorem provides an expression for $\frac{\partial R_{\rho_0}^{(p)}}{\partial p}(t)$ based on [3], thus extending [GAN 90] and adapting the classical works inspired by [LIO 68] to the probabilistic framework. The proof is not given here due to the reduced size of the present paper but will be provided in a forthcoming paper.

Theorem 3 *Under assumptions $\mathcal{H}$, the function $p \longmapsto R_{\rho_0}^{(p)}(t)$ is continuously differentiable with respect to p and*

$$\frac{\partial R_{\rho_0}^{(p)}}{\partial p}(t) = \int_0^t \rho_s^{(p)} \frac{\partial h^{(p)}}{\partial p}\,\mathrm{d}s - \int_0^t \rho_s^{(p)} \frac{\partial H^{(p)}}{\partial p} \varphi_t^{(p)} \mathrm{d}s \qquad [4]$$

where, for all $\varphi \in \mathcal{D}_H$ and all $(i,x,s) \in E \times V \times \mathbb{R}_+$, we set:

$$\frac{\partial H^{(p)}}{\partial p}\varphi(i,x,s) := \sum_{j \in E} \frac{\partial a^{(p)}}{\partial p}(i,j,x)\, \varphi(j, F^{(p)}(i,j,x), s) + \frac{\partial v^{(p)}}{\partial p}(i,x) \cdot \nabla\varphi(i,x,s)$$

$$+ \sum_{j \in E} a^{(p)}(i,j,x) \left(\nabla\varphi(j, F^{(p)}(i,j,x), s) \cdot \frac{\partial F^{(p)}}{\partial p}(i,j,x) \right).$$

3. The finite volume scheme

We now want to provide a numerical approximation for $\frac{\partial R_{\rho_0}^{(p)}}{\partial p}(t)$, writing a discrete counterpart of expression [4]. With that aim, we first calculate $\rho_s(i, \mathrm{d}x)$ using the implicit finite volume scheme from [EYM] (which is known to converge to the unique solution of [3]). The adjoint state method applied to the discrete setting will then also provide an approximation for $\varphi_t^{(p)}(i,x,s)$ for all $0 \le s \le t$ and all $(i,x) \in V$. Let us first recall the implicit finite volume scheme: let $\mathcal{M}$ be a given partition of V satisfying regularity properties (details in [EYM]) and for $N(p)$ fixed, let $\varepsilon \in [0, C_{\mathbf{v},N(p)}]$ be fixed too. We then set:

$$w_{K,L}^{(i)} = \max(|v_{K,L}^{(i)}|, \varepsilon) \text{ with } v_{K,L}^{(i)} = \frac{1}{\mathrm{m}(K|L)} \int_{K|L} \mathbf{v}^{(i)}(x) \cdot \mathbf{n}_{KL} \mathrm{d}s(x),$$

for all $K \in \mathcal{M}$, all $L \in \mathcal{N}_K$ and all $i \in E$, where $K|L$ stands for the interface between K and L, $\mathrm{d}s(x)$ is the $N-1$ dimensional measure on $K|L$, and $\mathrm{m}(K|L)$ is the measure of $K|L$.

Note: The condition $\varepsilon > 0$ is used in the convergence proof in [EYM]. Nevertheless, in the practical cases presented here, we let $\varepsilon = 0$ in order to avoid an increase of the numerical diffusion.

We also set:

$$a_{K,L}^{(i,j)} = \frac{1}{\mathrm{m}(K)} \int_K a^{(i,j)}(x) \left(\int_L \mu(i,j,x)(\mathrm{d}y) \right) \mathrm{d}x \qquad [5]$$

for all $K, L \in \mathcal{M}$, $i, j \in E$. For a given time step $k > 0$, the scheme then writes:

$$\rho_{i,K}^{(0)} = \tfrac{1}{\mathrm{m}(K)} \int_K \rho_0(i, \mathrm{d}x), \ \forall K \in \mathcal{M}, \forall i \in E$$

and

$$\begin{aligned}
&\mathrm{m}(K)(\rho_{i,K}^{(n+1)} - \rho_{i,K}^{(n)}) \\
&\quad + k \textstyle\sum_{L \in \mathcal{N}_K} \mathrm{m}(K|L) \left(v_{K,L}^{(i)} \frac{\rho_{i,K}^{(n+1)} + \rho_{i,L}^{(n+1)}}{2} + \frac{w_{K,L}^{(i)}}{2} (\rho_{i,K}^{(n+1)} - \rho_{i,L}^{(n+1)}) \right) \\
&\quad = -k\, \mathrm{m}(K) \rho_{i,K}^{(n+1)} \textstyle\sum_{j \in E} \sum_{L \in \mathcal{M}} a_{K,L}^{(i,j)} + k \sum_{j \in E} \sum_{L \in \mathcal{M}} \mathrm{m}(L) a_{L,K}^{(j,i)} \rho_{j,L}^{(n+1)},
\end{aligned} \qquad [6]$$

for all $K \in \mathcal{M}$, all $i \in E$ and all $n \in \mathbb{N}$.

For $(i, K) \in E \times \mathcal{M}$, we now set:

$$\bar{h}_{i,K} = \frac{1}{\mathrm{m}(K)} \int_K h^{(p)}(i, x) \, \mathrm{d}x. \qquad [7]$$

For $t = N\delta t$, with $N \geq 1$, a discrete approximation of $R_{\rho_0}^{(p)}(t)$ is given by:

$$\bar{R}_N^{(p)} = \sum_{n=1}^{N} \sum_{i \in E} \sum_{K \in \mathcal{M}} \delta t \, \mathrm{m}(K) \rho_{i,K}^{(n)} \bar{h}_{i,K}$$

By imitating the procedure used to evaluate $\frac{\partial}{\partial p} R_{\rho_0}^{(p)}(t)$ in [4], we introduce discrete versions of H and of the importance function φ_t (respectively $\bar{H}$ and $\bar{\varphi}$):

Lemma 4 *Let $D_{\bar{H}}$ be the set of the families $\theta = \left(\theta_{i,K}^{(n)}\right)_{(i,K,n) \in E \times \mathcal{M} \times \mathbb{N}}$ such that $\sup_{(i,K)} \left|\theta_{i,K}^{(n)}\right| < +\infty$ for all $n \in \mathbb{N}$, and let $\bar{H}$ be the operator defined on $D_{\bar{H}}$ by $\bar{H}(\theta) = \left(\bar{H}(\theta)_{i,K}^{(n)}\right)_{(i,K,n) \in E \times \mathcal{M} \times \mathbb{N}}$ with*

$$\begin{aligned}
\left(\bar{H}(\theta)\right)_{i,K}^{(n)} = {} & \frac{\theta_{i,K}^{(n+1)} - \theta_{i,K}^{(n)}}{\delta t} + \frac{1}{2\mathrm{m}(K)} \sum_{L \in \mathcal{N}_K} \left(\theta_{i,L}^{(n)} - \theta_{i,K}^{(n)}\right) \mathrm{m}(K|L) \left(v_{K,L}^{(i)} + w_{K,L}^{(i)}\right) \\
& + \sum_{j \in E} \sum_{L \in \mathcal{M}} a_{K,L}^{(i,j)} \left(\theta_{j,L}^{(n)} - \theta_{i,K}^{(n)}\right) \text{ for all } (i, K, n) \in E \times \mathcal{M} \times \mathbb{N}.
\end{aligned}$$

Given $\bar{h} = \left(\bar{h}_{i,K}\right)_{(i,K)\in E\times\mathcal{M}}$ *where* $\bar{h}_{i,K}$ *is defined by* [7]*, there exists a unique family* $\bar{\varphi} = \left(\bar{\varphi}_{i,K}^{(n)}\right)_{(i,K,n)\in E\times\mathcal{M}\times\mathbb{N}} \in D_{\bar{H}}$ *solution of:*

$$\left(\bar{H}\left(\bar{\varphi}\right)\right)_{i,K}^{(n)} = \bar{h}_{i,K} \quad \text{for all } (i,K,n) \in E\times\mathcal{M}\times\{0,...,N-1\} \qquad [8]$$

with $\bar{\varphi}_{i,K}^{(n)} = 0$ *for all* $(i,K) \in E\times\mathcal{M}$*, all* $n \geq N$*.*

We then have the following result, which gives an expression for $\frac{\partial}{\partial p}\left(\bar{R}_N^{(p)}\right)$.

Theorem 5 *The following relation holds:*

$$\frac{\partial}{\partial p}\left(\bar{R}_N^{(p)}\right) = \delta t \sum_{i\in E}\sum_{K\in\mathcal{M}}\sum_{n=1}^{N} \mathrm{m}(K)\rho_{i,K}^{(n)}\left(\frac{\partial}{\partial p}\bar{h}_{i,K} - \left(\frac{\partial\bar{H}}{\partial p}\right)_{i,K}^{(n-1)}(\bar{\varphi})\right) \qquad [9]$$

where, for all $(i,K,n) \in E\times\mathcal{M}\times\mathbb{N}$ *and* $\theta \in D_{\bar{H}}$*:*

$$\left(\frac{\partial\bar{H}}{\partial p}\right)_{i,K}^{(n)}(\theta) = \frac{1}{2\mathrm{m}(K)}\sum_{L\in\mathcal{N}_K}\left(\theta_{i,L}^{(n)} - \theta_{i,K}^{(n)}\right)\mathrm{m}(K|L)\frac{\partial}{\partial p}\left(v_{K,L}^{(i)} + w_{K,L}^{(i)}\right)$$
$$+ \sum_{j\in E}\sum_{L\in\mathcal{M}}\left(\frac{\partial}{\partial p}a_{K,L}^{(i,j)}\right)\left(\theta_{j,L}^{(n)} - \theta_{i,K}^{(n)}\right)$$

We show on the numerical example the classical properties resulting from the use of this adjoint method for the sensitivity analysis.

4. Numerical results

The following example is a simplified version of a benchmark studied in [DUF 06] and [EYM 08]. A gas production device is considered. It is composed of one production unit, which can be adjusted up or down. When up, the production rate of the unit varies between nominal and maximal rates, with nominal rate $\phi_{nom} = 7,500$ m^3/h and maximal rate $\phi_{max} = 10,000$ m^3/h. When down, the production rate of the unit is zero. The device is required to produce gas at the nominal rate ϕ_{nom}. In order to prevent the device production being stopped due to failures of the unit, a reservoir $\mathcal{R}$ is used, with maximal capacity $R = 2\times 10^6$ m^3: when the unit is down, the device production is achieved by taking in $\mathcal{R}$ the required production, at least as long as the level in $\mathcal{R}$ is not too low. When the unit is up, its production rate is nominal as long as $\mathcal{R}$ is full. When the level in $\mathcal{R}$ is lower, the unit produces at a higher rate (maximal rate, as long as the level in $\mathcal{R}$ is not too high) and the complementary production is used to refill $\mathcal{R}$. The device production rate is then a function of the unit state and of the level in $\mathcal{R}$. The repair time of the unit is log-normally distributed, with p.d.f.:

$f_{(t_0,\sigma)}(t) = \frac{1}{t\,\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln(t/t_0)}{\sigma}\right)^2\right)$ for $t > 0$,
where $t_0 = 1.26$ h, $\sigma = 2.25$ h, mean = 15.8 h, standard deviation = 198 h.
The associated hazard rate function is:
$a_{(t_0,\sigma)}(t) = f_{(t_0,\sigma)}(t) \,/\, \int_t^{+\infty} f_{(t_0,\sigma)}(s)\,\mathrm{d}s$ for $t > 0$.
The time to failure of the unit is Weibull distributed with the associated hazard rate function: $a_{(\alpha,\beta)} = \alpha\beta t^{\beta-1}$ for $t > 0$, where $\alpha = \frac{1}{10^3}$ h, $\beta = 1.01$, mean $= 930$ h, standard deviation $= 921$ h.
We set $E = \{0, 1\}$, where 0, 1 are the down and up states for the unit, respectively. The time-evolution of the system is then described by a PDMP $(I_t, X_t)_{t\geq 0}$ with values in $E \times \mathbb{R}^2$ where $X_t = (X_{1,t}, X_{2,t})$: component $X_{1,t}$ stands for the time elapsed in the current discrete state; component $X_{2,t}$ stands for the level in the reservoir. The initial state of the system is $(I_0, X_0) = (1, (0, R))$. The data for the PDMP are:

$$a(1,0,x) = \begin{cases} a_{(\alpha,\beta)}(x_1) & \text{if } x_1 \geq 0 \\ a_{(\alpha,\beta)}(0) = 0 & \text{if } x_1 < 0 \end{cases} \quad ; \quad a(0,1,x) = \begin{cases} a_{(t_0,\sigma)}(x_1) & \text{if } x_1 \geq 0 \\ a_{(t_0,\sigma)}(0) = 0 & \text{if } x_1 < 0 \end{cases}$$

$$\mu(1,0,x)(\mathrm{d}y) = \mu(0,1,x)(\mathrm{d}y) = \delta_{(0,x_2)}(\mathrm{d}y)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. We assume the speed of filling / emptying the reservoir to be as follows: we set $r_1 = r_0 = \frac{R}{10}$ and, for $x = (x_1, x_2) \in \mathbb{R}^2$, we take:

$$\mathbf{v}(1,x) = \left(1, (\phi_{max} - \phi_{nom}) \min\left(\frac{(R - x_2)^+}{r_1}, 1\right)\right)$$

$$\mathbf{v}(0,x) = \left(1, -\phi_{nom} \min\left(\frac{x_2^+}{r_0}, 1\right)\right)$$

The device production rate function is given by: $\phi(i,x) = \phi_{nom} h(i,x)$ with $h(1,x) = 1$ and $h(0,x) = \min\left(\frac{x_2^+}{r_0}, 1\right)$.
For this benchmark, we focus on the cumulative production availability at time $t = 100,000$ h, obtained by plugging the function h in [2], and on its derivative with respect of $p \in P$ with $P = \{t_0, \sigma, \alpha, \beta, \phi_{nom}, \phi_{max}, r_0, r_1, R\}$.
Since the discretization of $\mathbb{R}^2$ needs to be finite to implement the numerical scheme, we consider some gridded domain of the shape $[0, L_i] \times [0, R]$ (with $i = 0, 1$), where L_i is chosen large enough to ensure most of the probability mass will lie in $[0, L_i] \times [0, R]$: hence we take $L_0 = 2.5\ 10^6$ h and $L_1 = 1\ 10^6$ h, which leads to $\rho(\{i\} \times \mathbb{R}\backslash[0, L_i] \times [0, R]) \simeq 0$ for $i = 0, 1$ at the precision of the results displayed just below. In both states, the domain is gridded with 500×40 grid blocks; the steps are in geometric progression in the first direction, which corresponds to the repair/working times, and are constant in the second direction, which corresponds to the level of gas in the reservoir. The time step is equal to $\delta t = 1,000$ h. With these data, we find that $R_{\rho_0}(100,000) \simeq 99463.2$. The importance factors defined by [1] are approximated by [9], in which the derivatives of the coefficients are approximated by finite differences resulting from a variation ε of p. We then provide the value e, which is the relative error between the variation predicted by the importance factor,

and that which is observed by two simulations with a small variation of p. We also provide the importance factor for each parameter.

p	t_0	σ	α	β
e	$6.53E{-5}$	$2.43E{-4}$	$1.90E{-5}$	$1.18E{-4}$
IF_p	$-7.70E{-3}$	$-5.39E{-2}$	$-5.30E{-3}$	$-3.87E{-2}$

p	ϕ_{nom}	ϕ_{max}	r_0	r_1	R
e	$5.56E{-5}$	$4.75E{-9}$	$3.88E{-3}$	$4.97E{-3}$	$1.62E{-3}$
IF_p	$-2.71E{-3}$	$1.75E{-4}$	$-3.15E{-5}$	$-4.43E{-6}$	$2.55E{-3}$

We can see in these results that the relative error between the importance factors computed by the method presented here and by finite differences is very small, which shows the accuracy of the method. We can also notice that the most important parameters measured by their decreasing importance factors are σ, β, t_0, α, ϕ_{nom}, R, whereas $\phi_{\max}$, r_0 and r_1 seem somewhat less important. Finally, let us give the computing times: 19 s for the computation of both [6] and [8] and 10 s for [6] only. The computation of all derivatives hence requires $(9 + 1) \times 10 = 100$ s by finite differences, to be compared to the 19 s required by the method presented here, showing the efficiency of the method.

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