Sensitivity Estimates in Dynamic Reliability

Sophie MERCIER [1] and Michel ROUSSIGNOL
Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées, France

Abstract. The aim of this paper is to study and to compute first-order derivatives with respect to some parameter p, for some functionals of piecewise deterministic Markov processes (PDMP), in view of sensitivity analysis in dynamic reliability. Such functionals are mean values of some function of the process, cumulated on some finite interval $[0, t]$, and their asymptotic value per unit time.

Keywords. Piecewise deterministic Markov process, Importance factor.

1. Introduction

In dynamic reliability, the time-evolution of a system is described by a piecewise deterministic Markov process (PDMP) $(I_t, X_t)_{t\geq 0}$ (see [3], [5]). The first component I_t is discrete, with values in a finite state space E. Typically, it indicates the state (up/down) for each component of the system at time t. The second component X_t, with values in $V \subset \mathbb{R}^d$, stands for environmental conditions, such as temperature, pressure, and so on. Both components of the process, I_t and X_t, interact in each other: the process jumps at countably many isolated random times; by a jump from $(I_{t^-}, X_{t^-}) = (i, x)$ to $(I_t, X_t) = (j, y)$ (with (i, x), $(j, y) \in E \times V$), the transition rate between the discrete states i and j depends on the environmental condition x just before the jump and is a function $x \longmapsto a\,(i, j, x)$. Similarly, the environmental condition just after the jump X_t is distributed according to some distribution $\mu\,(i, j, x)\,(dy)$, which depends on both components just before the jump (i, x) and on the after jump discrete state j. Between jumps, the discrete component I_t is constant, whereas the evolution of the environmental condition X_t is deterministic, solution of a set of o.d.e. which depends on the fixed discrete state: given that $I_t = i$ for all $t \in [a, b]$, we have $\frac{d}{dt} X_t = \mathbf{v}(i, X_t)$ for all $t \in [a, b]$, where $\mathbf{v}$ is a mapping from $E \times V$ to V. Under technical assumptions, $(I_t, X_t)_{t\geq 0}$ is a Markov process with general state space $E \times V$ (see [3], [5]).

We study quantities of the form:

$$R_{\rho_0}(t) = \mathbb{E}_{\rho_0}\left(\int_0^t h(I_s, X_s)\, ds\right)$$

[1] Corresponding Author: Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées (CNRS UMR 8050), 5 boulevard Descartes, Champs sur Marne, F-77454 Marne-la-Vallée, France.
E-mail: sophie.mercier@univ-mlv.fr

where ρ_0 is the initial distribution of the process and h is some bounded measurable function.

We assume that the jump rates for I_t and the function h depend on some parameter p, where p belongs to an open set $O \subset \mathbb{R}$ (or $\mathbb{R}^k$). The quantities of interest then are the first-order derivatives of $R_{\rho_0}(t)$ and of $\lim_{t\to+\infty} R_{\rho_0}(t)/t$ with respect to p, which may help to rank input data according to their relative importance. This kind of sensitivity analysis was studied by Gandini [7] and by Cao and Chen [2] for pure jump Markov processes with countable state space. We here present extensions of their results to PDMPs. Note that due to the reduced size of the present paper, proofs are not provided here and will be part of a forthcomming paper.

All the paper is written under the following assumptions $(\mathcal{H}_1)$, where exponent $^{(p)}$ reflects dependence on p: for all $(i, j, p) \in E^2 \times O$, the function $x \longmapsto a^{(p)}(i, j, x)$ is non-negative, bounded and continuous; for all $(i, j, x) \in E^2 \times V$, the function $p \longmapsto a^{(p)}(i, j, x)$ is differentiable with uniformly bounded derivative for $(i, j, x, p) \in E^2 \times V \times O$; for all $i, j \in E$ and for all function $\psi \quad V \to \mathbb{R}$ continuous and bounded, the function $x \longmapsto \mu_{(i,j,x)}\psi = \int \psi(y)\mu_{(i,j,x)}(dy)$ is continuous; for all $i \in E$, the function $x \longmapsto \mathbf{v}(i, x)$ is locally Lipschitz continuous and sub-linear; for all $(i, x) \in E \times V$, the function $p \longmapsto h^{(p)}(i, x)$ is differentiable with uniformly bounded derivative for $(i, x, p) \in E^2 \times V \times O$.

We denote by $\rho_t^{(p)}(j, dy)$ the distribution of the process $\left(I_t^{(p)}, X_t^{(p)}\right)_{t\geq 0}$ at time t with initial distribution ρ_0 (independent on p). We then have:

$$R_{\rho_0}^{(p)}(t) = \int_0^t \rho_s^{(p)} h^{(p)}\, ds = \sum_{i\in E} \int_V \left(\int_0^t h^{(p)}(i, x)\, ds\right) \rho_s^{(p)}(i, dx).$$

2. Transitory results

We first introduce the infinitesimal generators of both Markov processes $\left(I_t^{(p)}, X_t^{(p)}\right)_{t\geq 0}$ and $\left(I_t^{(p)}, X_t^{(p)}, t\right)_{t\geq 0}$:

Definition 1 *Let $\mathcal{D}_{H_0}$ be the set of functions φ from $E \times V$ to $\mathbb{R}$ such that for all $i \in E$ the function $x \longmapsto \varphi(i, x)$ is bounded and continuously differentiable and the function $x \longmapsto \mathbf{v}(i, x) \cdot \nabla\varphi(i, x)$ is bounded and continuous on V. For $\varphi \in \mathcal{D}_{H_0}$, we define*

$$H_0^{(p)}\varphi(i, x) = \sum_{j\in E} a^{(p)}(i, j, x) \left(\mu_{(i,j,x)}\varphi(j, \cdot)\right) + \mathbf{v}(i, x) \cdot \nabla\varphi(i, x)$$

for all $(i, x) \in E \times V$, with $a^{(p)}(i, i, x) = -\sum_{j\neq i} a^{(p)}(i, j, x)$ and $\mu_{(i,i,x)}(dy) = \delta_x(dy)$, where δ_x is the Dirac measure at x. Let $\mathcal{D}_H$ be the set of functions φ from $E \times V \times \mathbb{R}$ to $\mathbb{R}$ such that for all $i \in E$ and $s \in \mathbb{R}_+$ the function $x \longmapsto \varphi(i, x, s)$ is bounded and continuously differentiable on V and the function $x \longmapsto \frac{\partial}{\partial s}\varphi(i, x, s) + \mathbf{v}(i, x) \cdot \nabla\varphi(i, x)$ is bounded and continuous on V. For $\varphi \in \mathcal{D}_H$, we define

$$H^{(p)}\varphi(i, x, s) = \sum_j a^{(p)}(i, j, x) \left(\mu_{(i,j,x)}\varphi(j, \cdot, s)\right) + \frac{\partial\varphi}{\partial s}(i, x, s) + \mathbf{v}(i, x) \cdot \nabla\varphi(i, x, s)$$

for all $(i, x, s) \in E \times V \times \mathbb{R}_+$.

Setting $P_t^{(p)}(i, x, j, dy)$ to be the transition probability distribution of $\left(I_t^{(p)}, X_t^{(p)}\right)_{t \geq 0}$, we then have: $P_s^{(p)}\varphi = \varphi + \int_0^s H_0^{(p)}\left(P_u^{(p)}\varphi\right) du$ for all $\varphi \in \mathcal{D}_{H_0}$, all $s \in \mathbb{R}_+$ and $P_s^{(p)}\varphi(\cdot, \cdot, s) = \varphi(\cdot, \cdot, 0) + \int_0^s H^{(p)}\left(P_u^{(p)}\varphi\right)(\cdot, \cdot, u)\ du$ for all $\varphi \in \mathcal{D}_H$, all $s \in \mathbb{R}_+$ (Chapman-Kolmogorov equations).

We may now introduce new functions called importance functions:

Definition 2 *Let t be fixed. We say that a function $\varphi_t^{(p)} \in \mathcal{D}_H$ is the importance function associated to $\left(h^{(p)}, t\right)$ if $\varphi_t^{(p)}$ is solution of the differential equation $H^{(p)}\varphi_t^{(p)}(i, x, s) = h^{(p)}(i, x)$ for all $s \in [0, t[$, all $(i, x) \in E \times V$ with initial data: $\varphi_t^{(p)}(i, x, t) = 0$ for all $(i, x) \in E \times V$.*

In applications, the importance functions will generally be computed numerically. However, an analytical form is available, which is also useful for the asymptotic study.

Lemma 3 *Let us assume that the function $x \longmapsto a^{(p)}(i, j, x)$ is continuously differentiable on V for all $i, j \in E$ and all $p \in O$, and that the function $\mathbf{v}$ is bounded (assumptions $\mathcal{H}_2$). The importance function associated to $\left(h^{(p)}, t\right)$ is then unique and it is given by:*

$$\varphi_t^{(p)}(i, x, s) = \begin{cases} -\int_0^{t-s}\left(P_u^{(p)}h^{(p)}\right)(i, x)\, du \text{ if } 0 \leq s \leq t \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

for all $(i, x) \in E \times V$.

The proof of the previous result is based on the Chapman-Kolmogorov equations, as well as the following theorem.

Theorem 4 *Let t be fixed. Under assumptions $\mathcal{H}_1$ and $\mathcal{H}_2$, the function $p \longmapsto R_{\rho_0}^{(p)}(t)$ is differentiable on O and we have:*

$$\frac{\partial R_{\rho_0}^{(p)}(t)}{\partial p} = \int_0^t \rho_s^{(p)} \frac{\partial h^{(p)}}{\partial p}\, ds + \int_0^t \rho_s^{(p)} \frac{\partial H^{(p)}}{\partial p} \varphi_{t-s}^{(p)} ds \tag{2}$$

where we set $\frac{\partial H^{(p)}}{\partial p}\varphi(i, x, s) = \sum_{j \in E} \frac{\partial a^{(p)}}{\partial p}(i, j, x)\left(\mu_{(i,j,x)}\varphi(j, \cdot, s)\right)$ for all $\varphi \in \mathcal{D}_H$, all $(i, x, s) \in E \times V \times \mathbb{R}_+$.

Equation (2) is an extension of the results from [7] for pure jump Markov processes.

3. Asymptotic results

We first transform (2) in view of studying its asymptotic expression.

Lemma 5 *Under assumptions $\mathcal{H}_1$ and $\mathcal{H}_2$, we have:*

$$\frac{1}{t}\frac{\partial R_{\rho_0}^{(p)}}{\partial p}(t)$$
$$=\frac{1}{t}\int_0^t \rho_s^{(p)}\frac{\partial h^{(p)}}{\partial p}\,ds+\frac{1}{t}\int_0^t \rho_s^{(p)}\frac{\partial H^{(p)}}{\partial p}\left(\int_0^{t-s}\left(P_u^{(p)}h^{(p)}-\pi^{(p)}h^{(p)}\right)\,du\right)ds$$

The proof of the previous lemma is based on $(1-2)$, and on the fact that $\frac{\partial H^{(p)}}{\partial p}\left(\pi^{(p)}h^{(p)}\right)=\left(\pi^{(p)}h^{(p)}\right)\frac{\partial H^{(p)}}{\partial p}\mathbf{1}=0$ because $H^{(p)}\mathbf{1}=0$.

We now need the following additional assumptions ($\mathcal{H}_3$): the process $(I_t, X_t)_{t\geq 0}$ is positive Harris-recurrent with $\pi^{(p)}$ as unique stationary distribution, and for each $p\in O$, there exists a function $f^{(p)}$ such that $\int_0^{+\infty} f^{(p)}(u)\,du<+\infty$, $\int_0^{+\infty} u\, f^{(p)}(u)\,du<+\infty$, $\lim_{u\to+\infty} f^{(p)}(u)=0$ and $\left|\left(P_u^{(p)}h^{(p)}\right)(i,x)-\pi^{(p)}h^{(p)}\right|\leq f^{(p)}(u)$ for all $(i,x)\in E\times V$, all $u\geq 0$. We get:

Theorem 6 *Under assumptions $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$, the function*

$$Uh^{(p)}(i,x)=\int_0^{+\infty}\left(\left(P_u^{(p)}h^{(p)}\right)(i,x)-\pi^{(p)}h^{(p)}\right)\,du$$

exists for all $(i,x)\in E\times V$ and

$$\lim_{t\to+\infty}\frac{1}{t}\frac{\partial R_{\rho_0}^{(p)}}{\partial p}(t)=\pi^{(p)}\frac{\partial h^{(p)}}{\partial p}+\pi^{(p)}\frac{\partial H_0^{(p)}}{\partial p}Uh^{(p)} \tag{3}$$

where we set $\frac{\partial H_0^{(p)}}{\partial p}\varphi(i,x)=\sum_{j\in E}\frac{\partial a^{(p)}}{\partial p}(i,j,x)\left(\mu_{(i,j,x)}\varphi(j,\cdot)\right)$ for all $\varphi\in\mathcal{D}_{H_0}$, all $(i,x)\in E\times V$.

Besides, the function $Uh^{(p)}$ is solution of the following differential equation:

$$H_0^{(p)}Uh^{(p)}(i,x)=\pi^{(p)}h^{(p)}-h^{(p)}(i,x)$$

The previous theorem provides an extension of the results from [2] for pure jump Markov processes.

We now look at two examples. In such examples, dependence on p (namely $^{(p)}$) is generally not specified any more, in order to get simpler notations.

4. A first example

A single component is considered, which is perfectly and instantaneously repaired at each failure. The time evolution of the component is described by the process $(X_t)_{t\geq 0}$ where X_t stands for the time elapsed at time t since the last instantaneous repair. (There is one single discrete state here so that component I_t is not necessary). The failure rate for the component at time t is $\lambda(X_t)$ where $\lambda(\cdot)$ is some continuous non negative function.

The process $(X_t)_{t\geq 0}$ is "renewed" after each repair so that $\mu(x)(dy) = \delta_0(dy)$ and the evolution of $(X_t)_{t\geq 0}$ between renewals is given by $g(x,t) = x + t$.

We are interested in the rate of renewals on $[0,t]$, namely in the quantity $Q(t)$ such that:

$$Q(t) = \frac{R(t)}{t} = \frac{1}{t}\mathbb{E}_0\left(\int_0^t \lambda(X_s)\ ds\right) = \frac{1}{t}\int_0^t \left(\int_{\mathbb{R}_+} \lambda(x)\rho_s(dx)\right) ds$$

where $R(t)$ is the renewal function associated to the underlying renewal process and ρ_s is the distribution of X_s given that $X_0 = 0$.

The function $\lambda(x)$ depends on some parameter p and we want to compute $\frac{\partial Q(t)}{\partial p}$. Using $(1-2)$, we get:

$$\frac{\partial Q(t)}{\partial p} = \frac{1}{t}\int_0^t \int_0^s \rho_s(dx)\frac{\partial \lambda}{\partial p}(x)\left(1 - \varphi_t(0,s) + \varphi_t(x,s)\right)\ ds$$

where φ_t is solution of $\lambda(x)\left(\varphi_t(0,s) - \varphi_t(x,s)\right) + \frac{\partial}{\partial s}\varphi_t(x,s) + \frac{\partial}{\partial x}\varphi_t(x,s) = \lambda(x)$ for all $s \in 0, t$ and $\varphi_t(x,t) = 0$ for all $x \in [0,t]$. No closed form is available for φ_t and for the numerical computation, this equation has been discretized and solved numerically.

For the asymptotic quantity, one may prove that:

$$\frac{\partial Q(\infty)}{\partial p} = Q(\infty)\int_0^{+\infty}\frac{\partial \lambda}{\partial p}(x)\left(1 - Q(\infty)\int_0^x e^{-\int_0^v \lambda(u)du}dv\right)dx$$

with $Q(\infty) = \frac{1}{\mathbb{E}(T_1)}$ and $\mathbb{E}(T_1) = \int_0^{+\infty} e^{-\int_0^v \lambda(u)du}dv$ (mean up time).

We take $\lambda(t) = \alpha\beta t^{\beta-1}$ and $(\alpha,\beta) = \left(10^{-5}, 4\right)$ and we compute $\frac{\partial Q(t)}{\partial p}$ for $t \leq \infty$ and $p = \alpha, \beta$. To validate our results, they are compared to those obtained by finite differences with:

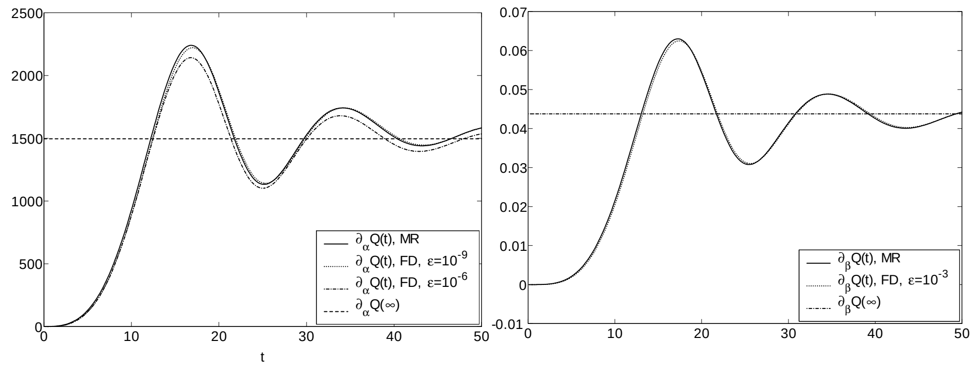
$$\frac{\partial Q(t)}{\partial p} \simeq \frac{1}{\varepsilon}\left(Q^{(p+\varepsilon)}(t) - Q^{(p)}(t)\right)$$

for small ε and $t \leq \infty$, with $p = \alpha, \beta$. For the asymptotic results, we use $Q(\infty) = \frac{1}{\mathbb{E}(T_1)}$ to compute such a derivative. For the transitory results, we use an algorithm from [8] which provides the renewal function $R(t)$ and hence $Q(t) = \frac{R(t)}{t}$.

The results are gathered in Table 1 for the asymptotic derivatives and plotted in Figures 1 and 2 for the transitory results, both by the present method and by finite differences, which are quite concordant (at least for ε small enough).

Table 1. $\frac{\partial Q(\infty)}{\partial \alpha}$ and $\frac{\partial Q(\infty)}{\partial \beta}$ by finite differences (FD) and the present method (MR)

	FD				MR
ε	10^{-4}	10^{-6}	10^{-8}	10^{-10}	
$\frac{\partial Q(\infty)}{\partial \alpha}$	5.1×10^2	1.496×10^3	1.5504×10^3	1.5510×10^3	1.5509×10^3
ε	10^{-4}	10^{-6}	10^{-8}	10^{-10}	
$\frac{\partial Q(\infty)}{\partial \beta}$	4.3761×10^{-2}	4.3760×10^{-2}	4.3760×10^{-2}	4.3760×10^{-2}	4.3755×10^{-2}



Figures 1 and 2. $\frac{\partial Q(t)}{\partial \alpha}$ and $\frac{\partial Q(t)}{\partial \beta}$ by FD and MR.

5. A second example

A tank is considered, which may be filled in or emptied out using a pump. This pump may be in two different states: "in" (state 0) or "out" (state 1). The level of liquid in the tank goes from 0 up to R. The state of the system "tank-pump" at time t is (I_t, X_t) where I_t is the discrete state of the pump ($I_t \in \{0, 1\}$) and X_t is the continuous level in the tank ($X_t \in [0, R]$). The transition rate from state 0 (resp. 1) to state 1 (resp. 0) at time t is $\lambda_0(X_t)$ (resp. $\lambda_1(X_t)$). The speed of variation for the liquid level in state 0 is $\mathbf{v}_0(x) = r_0(x)$ with $r_0(x) > 0$ for all $x \in [0, R[$ and $r_0(R) = 0$: the level increases in state 0 up to reaching R, where it remains constant. Similarly, the speed in state 1 is $\mathbf{v}_1(x) = -r_1(x)$ with $r_1(x) > 0$ for all $x \in]0, R]$ and $r_1(0) = 0$: the level of liquid decreases in state 1 until reaching 0, where it remains constant. Also, the level in the tank is continuous so that $\mu(i, 1-i, x)(dy) = \delta_x(dy)$ for $i \in \{0, 1\}$, all $x \in [0, R]$. The functions r_i and λ_i are assumed to be continuous, with λ_i bounded, which ensures an almost sure finite number of jumps on $[0, t]$ (all $t \geq 0$).

Such an example is very similar to that from [1]. The main difference is that we here assume X_t to remain bounded ($X_t \in [0, R]$) whereas X_t takes its values in $\mathbb{R}_+$ in the quoted paper.

In order to study asymptotic quantities, we assume conditions which ensures the process $(I_t, X_t)_{t\geq 0}$ to be φ-irreducible, in the sense of [6]. Such conditions for irreducibility are very similar to those from [1]: we first take $\lambda_1(0) > 0$ and $\lambda_0(R) > 0$ which prevents the system from being stuck in states $(1, 0)$ and $(0, R)$, respectively. Setting $t^{(i)}_{x\to y}$ for the deterministic time to go from x up to y following the curve $(g(i, x, t))_{t\in\mathbb{R}}$ (all $x, y \in [0, R]$), we also assume that:

$$\text{if } \int_x^R \frac{1}{r_0(u)} du = t^{(0)}_{x\to R} = +\infty, \text{ then } \int_x^R \frac{\lambda_0(u)}{r_0(u)} du = +\infty \text{ for some } x \in [0, R[$$

$$\text{if } \int_0^y \frac{1}{r_1(u)} du = t^{(1)}_{y\to 0} = +\infty, \text{ then } \int_0^y \frac{\lambda_1(u)}{r_1(u)} du = +\infty \text{ for some } y \in]0, R]$$

Under such conditions, the process $(I_t, X_t)_{t\geq 0}$ may be proved to be positive Harris recurrent and its single invariant distribution π is available in closed form.

We are interested in two quantities: first, the proportion of time spent by the level in the tank between two fixed bounds a and b with $0 < a < b < R$ and we set:

$$Q_0(t) = \frac{1}{t}\mathbb{E}_{\rho_0}\left(\int_0^t \mathbf{1}_{\{a \le X_s \le b\}} ds\right) = \frac{1}{t}\sum_{i=0}^{1}\int_0^t \int_a^b \rho_s(i, dx)\ ds = \frac{1}{t}\int_0^t \rho_s h_1\ ds$$

with $h_0(i, x) = \mathbf{1}_{[a,b]}(x)$.

Secondly, the mean number of times the pump is turned from state "in" (0) to state "out" (1) by unit time, namely:

$$Q_1(t) = \frac{1}{t}\mathbb{E}_{\rho_0}\left(\sum_{0<s\le t} \mathbf{1}_{\{I_{s-}=0; I_s=1\}}\right) = \frac{1}{t}\mathbb{E}_{\rho_0}\left(\int_0^t \lambda_0(X_s)\mathbf{1}_{\{I_s=0\}} ds\right) = \frac{1}{t}\int_0^t \rho_s h_1\ ds$$

with $h_1(i, x) = \mathbf{1}_{\{i=0\}}\lambda_0(x)$.

For $i_1 = 0, 1$, we assume that $\lambda_{i_1}(x)$ depends on some parameter α_{i_1} (but no other data depends on α_{i_1}). For $i_0, i_1 \in \{0, 1\}$, we then want to compute $\frac{\partial Q_{i_0}(t)}{\partial \alpha_{i_1}}$ and $\frac{\partial Q_{i_0}(\infty)}{\partial \alpha_{i_1}}$.

As for the asymptotic derivatives, one may prove that

$$Uh_{i_0}(1, x) - Uh_{i_0}(0, x) = -\int_0^x \left(\frac{u_{i_0}(1, z)}{r_1(z)} + \frac{u_{i_0}(0, z)}{r_0(z)}\right) e^{\int_x^z \left(\frac{\lambda_1(y)}{r_1(y)} - \frac{\lambda_0(y)}{r_0(y)}\right) dy}$$

and closed forms are then available for $\frac{\partial Q_{i_0}(\infty)}{\partial \alpha_{i_1}}$ ($i_0, i_1 \in \{0, 1\}$), using (3).

As for the transitory quantities, one needs to compute numerically quantities of the shape $\rho_t h$ as well as the importance functions $\varphi_t^{(i_0)}(i_1, \cdot, \cdot)$ (where $i_0, i_1 \in \{0, 1\}$). Both are computed using finite volume methods as in [4] and numerical approximations for $\frac{\partial Q_{i_0}(t)}{\partial \alpha_{i_1}}$ are derived using (2).

The system is assumed to initially be in state $(I_0, X_0) = (0, R/2)$. Besides, we take:

$$\lambda_0(x) = x^{\alpha_0}\ ;\ r_0(x) = (R - x)^{r_0}\ ;\ \lambda_1(x) = (R - x)^{\alpha_1}\ ;\ r_1(x) = x^{r_1}$$

for $x \in [0, R]$ with $\alpha_i > 0$ and $r_i > 1$ and the following numerical values:

$$\alpha_0 = 1.05; r_0 = 1.2; \alpha_1 = 1.10; r_1 = 1.1; R = 1; a = 0.3; b = 0.7.$$

Similarly as for the first method, we test our results using finite differences (FD). The results are rather stable choosing different values for ε and the results are provided for $\varepsilon = 10^{-6}$. The asymptotic results are given in Table 2 and the transitory ones in Table 3 for $t = 2$. The results are very similar by FD and MR both for asymptotic and transitory quantities, which clearly validate the method.

Table 2. $\frac{\partial Q_{i_0}(\infty)}{\partial \alpha_{i_1}}$ by finite differences (FD) and the present method (MR).

	$\frac{\partial Q_0(\infty)}{\partial \alpha_i}$		$\frac{\partial Q_1(\infty)}{\partial \alpha_i}$	
i	FD	MR	FD	MR
0	-1.4471×10^{-2}	-1.4469×10^{-2}	-5.5294×10^{-2}	-5.5303×10^{-2}
1	-1.7471×10^{-2}	-1.7469×10^{-2}	-4.9948×10^{-2}	-4.9946×10^{-2}

Table 3. $\frac{\partial Q_{i_0}(t)}{\partial \alpha_{i_1}}$ for $t = 2$ by finite differences (FD) and the present method (MR).

	$\frac{\partial Q_0(2)}{\partial \alpha_i}$		$\frac{\partial Q_1(2)}{\partial \alpha_i}$	
i	FD	MR	FD	MR
0	-4.7561×10^{-2}	-4.7580×10^{-2}	-8.6747×10^{-2}	-8.6599×10^{-2}
1	-4.5566×10^{-3}	-4.5166×10^{-3}	-2.7299×10^{-2}	-2.7370×10^{-2}

6. Conclusion

We have here presented extension of the results from [2] and [7] to PDMPs. The computation of $\frac{\partial R}{\partial p}(t)$ requires the computation of the distribution $(\rho_s)_{0 \le s \le t}$ and of the importance function $(\varphi_s)_{0 \le s \le t}$, which are independent on the choice of the parameter with respect of which we differentiate. This means that in case derivatives with respect to different parameters are needed, say $p_1, ..., p_m$, both $(\rho_s)_{0 \le s \le t}$ and $(\varphi_s)_{0 \le s \le t}$ only have to be computed once. On the contrary, evaluation of $\frac{\partial R}{\partial p_i}(t)$ for some $i = 1, ..., m$ by finite differences requires the computation of $(\rho_s)_{0 \le s \le t}$ for the initial parameters $p_1, ..., p_m$ but also for the same set of parameters with one single p_i substituted by $p_i + \varepsilon$ (for the fixed i). This means that evaluation of $\frac{\partial R}{\partial p_i}(t)$ for all $i = 1, ..., m$ requires computation of $(\rho_s)_{0 \le s \le t}$ for $m + 1$ set of parameters (the initial set + m other sets). Our method is then cheaper in computing times than finite differences.

This paper is a first step for the computation of derivatives of functionals of PDMP with respect to some parameter p. Indeed, the present study is restricted to the case where only the discrete transition rates depend on parameter p. Some additional mathematical work remains to be done to extend the results to more general cases. Also, some reflexion should be lead on to go through the numerical computation of the importance functions, in case of larger studies than the small examples of this paper.

Acknowledgement: the authors would like to thank Anne Barros, Christophe Bérenguer, Laurence Dieulle and Antoine Grall from Troyes Technological University (Université Technologique de Troyes) for having drawn their attention to the present subject.

References

[1] O. Boxma, H. Kaspi, O. Kella, D. Perry (2005). On/Off Storage Systems with State Dependent Input, Output and Switching Rates. *Probab. Engrg. Inform. Sci.*, **19**(1), pp. 1–14.

[2] X.-R. Cao and H.-F. Chen (1997). Perturbation realization, potentials, and sensitivity analysis of Markov processes. *IEEE Trans. Automat. Contr.*, **42**(10), pp. 1382–1393.

[3] C. Cocozza-Thivent, R. Eymard, S. Mercier, M. Roussignol (2006). Characterization of the marginal distributions of Markov processes used in dynamic reliability. *J. Appl. Math. Stoch. Anal.* **2006**, pp. 1–18.

[4] C. Cocozza-Thivent, R. Eymard, S. Mercier (2006). A finite volume scheme for dynamic reliability models. *IMA J. Numer. Anal.*, **26**(3), pp. 446–471, Advance Access published on 6 March 2006.

[5] M. H. A. Davis (1993). *Markov models and optimization*. Monographs on Statistics and Applied Probability, Chapman and Hall, London.

[6] D. Down, S.P. Meyn and R. Tweedie (1996) Exponential and Uniform Ergodicity of Markov Processes. *Ann. Probab.*, **23**, pp. 1671–1691.

[7] A. Gandini (1990). Importance and sensitivity analysis in assessing system reliability. *IEEE Trans. Reliab.*, **39**(1), pp. 61–70.

[8] S. Mercier (2007). Discrete random bounds for general random variables and applications to reliability. *European J. Oper. Res.*, **177**(1), pp. 378–405, available online 15 Feb. 2006.