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Weak entropy solutions for degenerate parabolic–hyperbolic inequalities

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Abstract

We study inner obstacle problems for a class of *strongly degenerate* parabolic–hyperbolic quasilinear operators associated with Dirichlet data in an open bounded subset of $\mathbb{R}^p$, $p \geq 1$. We first give the definition of a *weak entropy solution* that warrants uniqueness; the boundary conditions are expressed by using the framework of *divergence measure fields*. The existence of such a solution is obtained through the vanishing viscosity method.

Keywords: Parabolic degenerate inequalities; Obstacle problems; Divergence measure fields

1. Introduction

1.1. Mathematical setting

Obstacle problems in physics and mechanics have been described and studied by many authors ([1–3], and so on). This paper focuses on the mathematical analysis of a positiveness condition for the quasilinear second-order operator stemming from the theory of fluid flows through porous media:

$$\mathbb{P}(t, x, .) : u \to \partial_t u + \sum_{i=1}^{p} \partial_{x_i} \chi_i(t, x, u) + \psi(t, x, u) - \Delta\phi(u),$$

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where ϕ is a nondecreasing function (in particular, ϕ' may be equal to zero on non-empty intervals of $\mathbb{R}$). Such a study within the context of petroleum engineering and for transport of pollutants in the subsoil has been developed in [10]. Let T be a positive real, Ω a bounded subset of $\mathbb{R}^p$, $p \geq 1$, $Q =]0, T[\times\Omega$ and $\Sigma =]0, T[\times\partial\Omega$; the outer normal of Ω is denoted as ν. For a given nonnegative measurable and bounded function u_0 we prove that the formal Cauchy–Dirichlet problem: find a bounded and measurable function u such that

$$u \geq 0 \text{ in } Q,\ \mathbb{P}(t, x, u) \geq 0 \text{ and } u\mathbb{P}(t, x, u) = 0 \text{ on } Q, \tag{1}$$

$$u = 0 \text{ on } \Sigma, \tag{2}$$

$$u(0, .) = u_0 \text{ on } \Omega, \tag{3}$$

has a unique solution. The special framework of a *strongly degenerate* operator $\mathbb{P}$ leads us to look for a weak entropy formulation for (1)–(3) in the same spirit as Carrillo [4] or more recently as Mascia, et al. [5], for some diffusion–convection equations. Such a formulation is motivated by the existence, in the computational domain, of nondegenerate parabolic zones (corresponding to $\phi' > 0$) and hyperbolic ones (in which $\phi' \equiv 0$), glued together in a way that depends on the solution itself. Moreover, as clearly mentioned in [5], in order to take into account possible boundary layers, the boundary conditions should be interpreted as compatibility inequalities on Σ, as they are in the case of quasilinear first-order equations (see [6] in the case of $BV(Q)\cap L^\infty(Q)$-solutions or [7] for only $L^\infty(Q)$-solutions). Here, we use the mathematical framework of divergence measure fields to provide a formulation that generalizes F. Otto's first-order relations to the second order.

1.2. *Notation and main assumptions on data*

The hypotheses on $\boldsymbol{\chi}$ and ψ are detailed in [8]. We simply mention that $\boldsymbol{\chi} \equiv (\chi_1, \ldots, \chi_p)$ and ψ have partial derivatives respectively to the second and first order and to deal with bounded solutions we suppose that $\partial_{x_i}\chi_i$ and ψ are Lipschitzian with respect to their third variable, uniformly in (t, x), with Lipschitz constants $M'_{\partial_{x_i}\chi_i}$ and M'_ψ. We thus define, for any t of $[0, T]$,

$$M(t) = \frac{K_1}{K_2}(e^{K_1 t} - 1) + \|u_0\|_{L^\infty(\Omega)} e^{K_1 t},$$

where $K_1 = \sum_{i\in\{1,\ldots,p\}} M'_{\partial_i\chi_i} + M'_\psi$ and $K_2 = \|Div_x \boldsymbol{\chi}(t, x, 0) + \psi(t, x, 0)\|_\infty$.

- $\phi \in W^{1,+\infty}(] - M(T), M(T)[)$ and $\phi(0) = 0$. Moreover, we set $E = \{l \in \mathbb{R}, \{l\} = \phi^{-1}\{\phi(l)\}\}$.
- $\partial\Omega$ is a $\mathcal{C}^2$-class frontier and is locally the graph of a $\mathcal{C}^2$-class function through a $\mathcal{C}^2$-covering with open sets $(B_i)_{i\in I}$, $I \subsetneq \mathbb{N}$. To simplify, we write $B \in \mathcal{B}$ where $\mathcal{B}$ is the set of all possible recoverings of $\partial\Omega$ (see [5]).
- For any n in $\mathbb{N}^*$, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure.
- $\mathcal{DM}_2(Q) = \{V \in (L^2(Q))^{p+1}, Div_{(t,x)}V \in \mathcal{M}_b(Q)\}$, where $\mathcal{M}_b(Q)$ is the space of bounded Radon measures on Q, is the L^2-space of the divergence measure field. The next generalized Gauss–Green formula coming from the one stated in [9] holds for any V in $\mathcal{DM}_2(Q)$ and ξ in $H^1(Q) \cap L^\infty(Q) \cap \mathcal{C}(Q)$:

$$\langle V, \xi\rangle_{\partial Q} = \int_Q V.(\partial_t\xi, \nabla\xi)\,\mathrm{d}x\mathrm{d}t + \int_Q \xi\,\mathrm{d}[Div_{(t,x)}V].$$

- $\forall\lambda > 0,\ \forall x \in \mathbb{R}^+,\ sgn_\lambda(x) = \min\left(\frac{x}{\lambda}, 1\right)$ and $sgn_\lambda(-x) = -sgn_\lambda(x)$.

2. Mathematical formulation and uniqueness property

Definition 1. A measurable bounded function u is a weak entropy solution to (1)–(3) if

$$u \geq 0 \text{ a.e. in } Q,\ \partial_t u \in L^2(0,T;H^{-1}(\Omega)), \phi(u) \in L^2(0,T;H_0^1(\Omega)), \tag{4}$$

$$ess \lim_{t\to 0^+} \int_\Omega |u(t,x) - u_0(x)|\,\mathrm{d}x = 0,$$

$$\forall k \in \mathbb{R}^+, \forall B \in \mathcal{B}, \forall \zeta \in \mathcal{D}(B), \zeta \geq 0,\ U_k\zeta \in \mathcal{DM}_2(Q), \tag{5}$$

$$\forall k \in \mathbb{R}^+, \forall \xi \in H_0^1(Q) \cap L^\infty(Q), \xi \geq 0, \int_Q U_k.\overline{\nabla}\xi\,\mathrm{d}x\mathrm{d}t - \int_Q sgn(u-k)G(u,k)\xi\,\mathrm{d}x\mathrm{d}t \geq 0, \tag{6}$$

$$\forall B \in \mathcal{B}, \forall \zeta \in \mathcal{D}(B), \zeta \geq 0, \int_\Sigma \boldsymbol{F}(k,0).\nu\xi\zeta\, d\mathcal{H}^p \leq \langle U_k\zeta, \xi\rangle_{\partial Q} + \langle U_0\zeta, \xi\rangle_{\partial Q}, \tag{7}$$

$\forall \xi \in L^\infty(Q) \cap H^1(Q) \cap \mathcal{C}(Q), \xi(T,.) = \xi(0,.) = 0, \xi \geq 0$ and $\forall k \in \mathbb{R}^+$ where

$$\boldsymbol{F}(u,k) = sgn(u-k)\{\boldsymbol{\chi}(t,x,u) - \boldsymbol{\chi}(t,x,k)\},\ G(u,k) = Div_x\boldsymbol{\chi}(t,x,k) + \psi(t,x,u),$$
$$U_k = (|u-k|, -\nabla|\phi(u) - \phi(k)| + \boldsymbol{F}(u,k)),\ \overline{\nabla}\zeta = (\partial_t\zeta, \nabla\zeta).$$

Remark 1. If u is a weak entropy solution to (1)–(3) then it is a weak solution in the sense that (4) holds and the strong variational inequality is fulfilled $\forall v \in H_0^1(\Omega)$, $v \geq 0$ a.e. in Ω, for a.e. t of $]0,T[$:

$$\langle\partial_t u, v - \phi(u)\rangle + \int_\Omega (\nabla\phi(u) - \boldsymbol{\chi}(t,x,u)).\nabla(v - \phi(u))\,\mathrm{d}x$$
$$+ \int_\Omega \psi(t,x,u)(v - \phi(u))\,\mathrm{d}x \geq 0. \tag{8}$$

We first establish the uniqueness of a weak entropy solution. The proof uses a comparison theorem which is a J. Carrillo extension to second-order equations of the classical hyperbolic method based on a doubling of the time and space variables [4]. For the treatment of the boundary terms the demonstration refers to [5]. However, numerous adaptations are necessary due to the framework of obstacle problems and the argumentation relies on two lemmas. The first one is an *inequality version* of the standard *energy equality* owing to Carrillo [4] and is satisfied by any weak solution:

Lemma 1. *Let u be a weak solution to* (1)–(3)*. Then,* $\forall \xi \in \mathcal{D}(Q), \xi \geq 0, \forall k \in E, k \geq 0,$

$$\int_Q (U_k.\overline{\nabla}\xi - sgn(u-k)G(u,k)\xi)\,\mathrm{d}x\mathrm{d}t \geq \limsup_{\lambda\to 0^+} \int_Q sgn'_\lambda(\phi(u) - \phi(k))(\nabla\phi(u))^2\xi\,\mathrm{d}x\mathrm{d}t.$$

Proof. We may choose $\phi(u) - \lambda/\|\xi\|_\infty\, sgn_\lambda(\phi(u) - \phi(k))\xi$ as a test function in (8). By integrating over $]0,T[$ we obtain an inequality in which the convective term is integrated by parts in order to pass to the limit with λ. By referring to the hypo-inverse ϕ_0^{-1} of ϕ and denoting

$$\mathbf{H}_\lambda(t,x,r) = \int_{\phi(k)}^r [\boldsymbol{\chi}(t,x,\phi_0^{-1}(\tau)) - \boldsymbol{\chi}(t,x,k)]sgn'_\lambda(\tau - \phi(k))\,\mathrm{d}\tau$$

we have

$$\int_Q (\boldsymbol{\chi}(t,x,u) - \boldsymbol{\chi}(t,x,k)).\nabla\phi(u)sgn'_\lambda(\phi(u) - \phi(k))\xi\,\mathrm{d}x\mathrm{d}t$$

$$= \int_Q Div_x \mathbf{H}_\lambda(t, x, \phi(u))\xi \, dx dt - \mathcal{O}_\lambda, \tag{9}$$

where in the right-hand side of (9) the first integral is integrated by parts and

$$\mathcal{O}_\lambda = \int_Q \int_{\phi(k)}^{\phi(u)} (Div_x \boldsymbol{\chi}(t, x, \phi_0^{-1}(\tau)) - Div_x \boldsymbol{\chi}(t, x, k))\, sgn'_\lambda(\tau - \phi(k))\, d\tau \xi \, dx dt.$$

Now let us come back to the definition of sgn'_λ and stress that, since k belongs to E, the generalized function ϕ_0^{-1} is continuous at $\phi(k)$; therefore the right-hand side of (9) goes to zero with λ. □

This energy inequality is not sufficient for proving uniqueness: it is fulfilled by any weak solution and is only true for k in E, $k \geq 0$. So we complement it with the inner entropy inequality (6), which is available for any k in $\mathbb{R}^+$. This technique, adapted from Carrillo's [4], leads to a Kruskov-type relation between two weak entropy solutions. Let Ψ be a nonnegative function of $\mathcal{D}(Q) \times \mathcal{D}(Q)$. We set $\bar{d} = dx dt d\tilde{x} d\tilde{t}$ and add a "tilde" superscript to any function in "tilde" variables.

Lemma 2. *If u_1 and u_2 are bounded measurable functions satisfying* (4) *and* (6), *then*

$$- \int_{Q\times Q} \{|u_1 - \tilde{u}_2|(\Psi_t + \Psi_{\tilde{t}}) + sgn(\phi(u_1) - \phi(\tilde{u}_2))(\nabla_x \phi(u_1) - \nabla_{\tilde{x}} \phi(\tilde{u}_2)).(\nabla_x \Psi + \nabla_{\tilde{x}} \Psi)\} \bar{d}$$
$$- \int_{Q\times Q} \left\{ \boldsymbol{F}(u_1, \tilde{u}_2).\nabla_x \Psi + \tilde{\boldsymbol{F}}(\tilde{u}_2, u_1).\nabla_{\tilde{x}} \Psi \right\} \bar{d} + \int_{Q\times Q} sgn(u_1 - \tilde{u}_2)(G(u_1, \tilde{u}_2)$$
$$- \tilde{G}(\tilde{u}_2, u_1))\Psi \bar{d} \leq 0.$$

Proof. On the one hand, in Lemma 1 written in variables (t, x) for u_1, we choose $k = u_2(\tilde{t}, \tilde{x})$ for a.e. $(\tilde{t}, \tilde{x})$ in $Q_0^{\tilde{u}_2} = \{(\tilde{t}, \tilde{x}) \in Q,\ u_2(\tilde{t}, \tilde{x}) \in E\}$. On the other hand, in (6) written in variables (t, x) for u_1, we choose $k = \tilde{u}_2(\tilde{t}, \tilde{x})$ for a.e. $(\tilde{t}, \tilde{x}) \in Q \setminus Q_0^{\tilde{u}_2}$. Each inequality obtained in this way is integrated with respect to $\tilde{t}$ and $\tilde{x}$ on the corresponding domain. By adding we obtain for u_1

$$\int_{Q\times Q} \left(U_{\tilde{u}_2}.\overline{\nabla}_{(t,x)} \Psi - sgn(u_1 - \tilde{u}_2) G(u_1, \tilde{u}_2) \Psi\right) \bar{d}$$
$$\geq \limsup_{\lambda\to 0^+} \int_{Q\times Q_0^{\tilde{u}_2}} sgn'_\lambda(\phi(u_1) - \phi(\tilde{u}_2))(\nabla\phi(u_1))^2 \Psi \bar{d}$$
$$\geq \limsup_{\lambda\to 0^+} \int_{Q_0^{u_1}\times Q_0^{\tilde{u}_2}} sgn'_\lambda(\phi(u_1) - \phi(\tilde{u}_2))(\nabla\phi(u_1))^2 \Psi \bar{d},$$

the last inequality being given by the fact that $\nabla\phi(u_1) = 0$ a.e. on $Q \setminus Q_0^{u_1}$.
Moreover, we integrate over Q the Gauss–Green formula:

$$\int_Q \nabla_x \phi(u_1).\nabla_{\tilde{x}}[sgn_\lambda(\phi(u_1) - \phi(\tilde{u}_2))\Psi]\, d\tilde{x} d\tilde{t} = 0.$$

We develop the partial derivatives and, since $\phi(\tilde{u}_2)$ belongs to $L^2(0, T; H_0^1(\Omega))$, the λ-limit provides

$$\int_{Q\times Q} \nabla_x |\phi(u_1) - \phi(\tilde{u}_2)|.\nabla_{\tilde{x}} \Psi \, \bar{d} = \lim_{\lambda\to 0^+} \int_{Q_0^{u_1}\times Q_0^{\tilde{u}_2}} sgn'_\lambda(\phi(u_1) - \phi(\tilde{u}_2))\nabla_x \phi(u_1).\nabla_{\tilde{x}} \phi(\tilde{u}_2)\Psi \, \bar{d}.$$

We apply the same reasoning for $\tilde{u}_2$ and group all the results to obtain the desired inequality. □

Now following [5], we state the T-Lipschitzian dependence in $L^1(\Omega)$:

Theorem 1. *The degenerate problem* (1)–(3) *admits at most one weak entropy solution. Moreover, if u_1 and u_2 are two weak entropy solutions associated with $u_{0,1}$ and $u_{0,2}$,*

$$\text{for a.e. } t \text{ in }]0,T[,\quad \int_\Omega |u_1(t,x)-u_2(t,x)|\,\mathrm{d}x \le e^{M'_\psi t}\int_\Omega |u_{0,1}(x)-u_{0,2}(x)|\,\mathrm{d}x.$$

3. Existence result

Let us now establish the existence of a *weak entropy* solution to (1)–(3) through the vanishing viscosity method. The latter consists in introducing some diffusion in the whole domain via a positive parameter δ destined to tend to 0^+. Then, we define $\phi_\delta = \phi + \delta Id_{\mathbb{R}}$, a bi-Lipschitzian function, so as to obtain the nondegenerate parabolic operator $\mathbb{P}_\delta$ and the corresponding unilateral obstacle problem formally described by: find a measurable and bounded function u_δ such that

$$u_\delta \ge 0 \text{ a.e. in } Q,\ \mathbb{P}_\delta(t,x,u_\delta) \ge 0 \text{ and } u_\delta \mathbb{P}_\delta(t,x,u_\delta) = 0 \text{ on } Q, \tag{10}$$

$$u_\delta = 0 \text{ on } \Sigma. \tag{11}$$

3.1. A regularization of the initial data

We look for a priori estimates of the sequence $(u_\delta)_{\delta>0}$ that are sufficient for specifying its behaviour when δ goes to 0^+. We seek Hilbertian estimates for $\phi_\delta(u_\delta)$ and $W^{1,1}(Q)$-estimates for u_δ. This requires smoothness assumptions on the gradient and on the Laplacian of ϕ_δ of the initial datum for (10) and (11). That is why we first introduce a regularization u_0^ϵ of u_0 obtained by means of mollifiers, so that

$$u_0^\epsilon \in \mathcal{D}(\Omega),\ u_0^\epsilon \ge 0 \text{ a.e. in } \Omega,\ \|u_0^\epsilon\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)},$$
$$\lim_{\epsilon\to 0^+} u_0^\epsilon = u_0 \text{ in } L^q(\Omega),\ 1 \le q < +\infty, \text{ and a.e. on } \Omega,$$

and secondly we consider for any positive μ and δ the solution $u_0^{\mu,\delta,\epsilon}$ of the problem

$$u_0^{\mu,\delta,\epsilon} - \mu\Delta\phi_\delta(u_0^{\mu,\delta,\epsilon}) = u_0^\epsilon \text{ in } \Omega,\ u_0^{\mu,\delta,\epsilon} = 0 \text{ on } \partial\Omega.$$

In that way,

Lemma 3. *$u_0^{\mu,\delta,\epsilon} \in H_0^1(\Omega)\cap L^\infty(\Omega)$, $\phi_\delta(u_0^{\mu,\delta,\epsilon}) \in H^2(\Omega)$ and $u_0^{\mu,\delta,\epsilon} \ge 0$ a.e. in Ω. Moreover, $\exists C > 0$ independent from δ, μ and ϵ such that*

$$\|u_0^{\mu,\delta,\epsilon}\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)},\ \mu\|\phi_\delta(u_0^{\mu,\delta,\epsilon})\|^2_{H_0^1(\Omega)} \le C,\ \|\nabla u_0^{\mu,\delta,\epsilon}\|_{L^1(\Omega)^p} \le C + \|\nabla u_0^\epsilon\|_{L^1(\Omega)^p}.$$

3.2. A priori estimates

Firstly we freeze ϵ and μ. To simplify the writing, they will be dropped as indexes. In this context, we first recall the property obtained in [10] by using the method of penalization:

Theorem 2. *For a given $u_0^{\mu,\delta,\epsilon}$, the problem* (10) *and* (11) *has a unique solution u_δ in $L^\infty(Q)\cap H^1(Q)\cap L^\infty(0,T;H_0^1(\Omega))\cap \mathcal{C}^0([0,T];L^q(\Omega))$, $1\le q<+\infty$, with $\phi_\delta(u_\delta)$ in $L^\infty(0,T;H_0^1(\Omega))$. Furthermore, u_δ is characterized through the strong variational inequality, for all v in $L^2(\Omega)$, $v\ge 0$,* *and a.e. on* $]0, T[$,

$$\int_{\Omega} \mathbb{P}_\delta(t, x, u_\delta)(v - \phi_\delta(u_\delta))\, \mathrm{d}x \geq 0,$$

and fulfils the a priori estimates:

$$\forall t \in [0, T],\ |u_\delta(t, .)| \leq M(t) \text{ a.e. in } \Omega,$$
$$\|\partial_t u_\delta\|_{L^2(0,T;H^{-1}(\Omega))} + \|F_\delta(u_\delta)\|_{L^2(0,T;H_0^1(\Omega))} \leq C_1,$$
$$\forall s \in [0, T],\ \|\partial_t F_\delta(u_\delta)\|^2_{L^2(Q_s)} + \|\phi_\delta(u_\delta)(s, .)\|^2_{H_0^1(\Omega)} \leq C_2 + \|\phi_\delta(u_0^{\mu,\delta,\epsilon})\|^2_{H_0^1(\Omega)},$$
$$\|\partial_t u_\delta\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla u_\delta\|_{L^\infty(0,T;L^1(\Omega)^p)} \leq A_1 + A_2(\|\nabla u_0^{\mu,\delta,\epsilon}\|_{L^1(\Omega)^p} + \|\Delta\phi_\delta(u_0^{\mu,\delta,\epsilon})\|_{L^1(\Omega)}),$$
$$\frac{1}{h}\|u_\delta(t + h, .) - u_\delta(t, .)\|_{L^1(\Omega)} \leq A_3 + A_4(\|\nabla u_0^{\mu,\delta,\epsilon}\|_{L^1(\Omega)^p} + \|\Delta\phi_\delta(u_0^{\mu,\delta,\epsilon})\|_{L^1(\Omega)}),$$

$\forall h \in]0, T[, \forall t \in]0, T - h[$, *where* C_i *and* A_i *are positive constants independent from any parameter and*

$$F_\delta(x) = \int_0^x (\phi'_\delta(\tau))^{1/2}\, \mathrm{d}\tau.$$

3.3. *The degenerate problem: existence of a weak entropy solution*

A priori estimates of Lemma 3 and a compactness argument ensure that as δ goes to $0^+ (\mathrm{u}_0^{\mu,\delta,\epsilon})_{\delta>0} L^q(\Omega)$-converges, $1 \leq q < +\infty$, toward $u_0^{\mu,\epsilon} \in BV(\Omega) \cap L^\infty(\Omega)$, the weak entropy solution in the sense of [4] or [5] to the degenerate elliptic problem

$$u_0^{\mu,\epsilon} - \mu\Delta\phi(u_0^{\mu,\epsilon}) = u_0^\epsilon \text{ in } \Omega,\ \phi(u_0^{\mu,\epsilon}) = 0 \text{ on } \partial\Omega.$$

Furthermore, $\exists C(\epsilon) > 0$ such that $\|u_0^{\mu,\epsilon}\|_{BV(\Omega)\cap L^\infty(\Omega)} \leq C(\epsilon)$. Besides this, Theorem 2 (with Lemma 3) ensures that $(u_\delta)_{\delta>0}$ remains in a fixed bounded subset of $W^{1,1}(Q) \cap L^\infty(Q)$. Thus, a compactness argument and Ascoli's lemma prove the existence of a function u in $BV(Q) \cap L^\infty(Q) \cap \mathcal{C}^0([0, T], L^1(\Omega))$ with $\partial_t u \in L^2(0, T; H_0^1(\Omega))$ satisfying $u \geq 0$ a.e. in Q and such that up to a subsequence, when $\delta \to 0^+$,

$$u_\delta \to u \text{ in } \mathcal{C}^0([0, T]; L^q(\Omega)), 1 \leq q < +\infty,$$
$$\phi_\delta(u_\delta) \rightharpoonup \phi(u) \text{ in } H^1(Q) \text{ weak.}$$

Therefore we can state:

Theorem 3. *For* μ *and* ϵ *fixed, the degenerate obstacle problem* (1) *and* (2) *admits a unique weak entropy solution* $u_{\mu,\epsilon}$ *associated with* $u_0^{\mu,\epsilon}$. *This solution belongs to* $BV(Q)\cap L^\infty(Q)\cap\mathcal{C}([0, T]; L^1(\Omega))$ *and is the limit of the whole sequence* $(u_\delta)_{\delta>0}$ *of solutions to problems* $((10), (11))_{\delta>0}$ *– with initial data* $(u_0^{\mu,\delta,\epsilon})_{\delta>0}$ *– in* $L^q(Q)$, *in* $\mathcal{C}([0, T]; L^q(\Omega))$, $1 \leq q < +\infty$, *and a.e. on* Q.

Idea of the proof. The key point is the proof of (5) whose demonstration is inspired by the one presented in [5], by coming back to the penalized problem associated with (10) and (11), which consists in introducing a positive parameter η and the nondegenerate parabolic operator $\mathbb{P}_{\delta,\eta}(t, x, .) : u \to \mathbb{P}_\delta(t, x, u) - u^-/\eta$. The convergence properties of $(u_{\delta,\eta})_{\eta>0}$ toward u_δ, as η goes to 0^+, are widely described in [10]. We take the $L^2(Q)$-scalar product between the viscous-penalized equation fulfilled by

$u_{\delta,\eta}$ and $sgn_\lambda(\phi_\delta(u_{\delta,\eta})-\phi_\delta(k))\zeta\xi$ where ξ belongs to $\mathcal{C}^\infty(\overline{Q})$ and $k \geq 0$. Accordingly, by passing to the limit with respect to λ we ensure the existence of a nonnegative $\kappa_{\delta,\eta}$ in $\mathcal{C}'(\overline{Q})$ (involving the penalized term) such that for any ξ of $\mathcal{C}^\infty(\overline{Q})$

$$\begin{aligned}\langle \kappa_{\delta,\eta}, \xi\rangle = &\int_Q U_k^{\delta,\eta}.\overline{\nabla}(\zeta\xi)\,\mathrm{d}x\mathrm{d}t - \int_\Sigma \mathbf{F}(0,k).\nu\zeta\xi\mathrm{d}\mathcal{H}^p - \int_Q G(u_{\delta,\eta},k)sgn(u_{\delta,\eta}-k)\zeta\xi\,\mathrm{d}x\mathrm{d}t\\ &- \int_\Omega |u_{\delta,\eta}(T,.)-k|\zeta\xi(T)\,\mathrm{d}x + \int_\Omega |u_0^{\mu,\delta,\epsilon}-k|\zeta\xi(0)\,\mathrm{d}x - sgn(k)J_{\delta,\eta}(\zeta\xi)\\ &+ \int_\Sigma \{\boldsymbol{\chi}(t,\sigma,0)-\boldsymbol{\chi}(t,\sigma,k)\}.\nu\zeta\xi\mathrm{d}\mathcal{H}^p. \end{aligned} \tag{12}$$

The term including the normal derivative of $\phi_\delta(u_{\delta,\eta})$ has been expressed by taking the $L^2(Q)$-scalar product between the viscous-penalized equation and $\zeta\xi$, thus leading to

$$\begin{aligned}J_{\delta,\eta}(\zeta\xi) = &-\int_Q u_{\delta,\eta}\zeta\partial_t\xi\,\mathrm{d}x\mathrm{d}t + \int_\Omega u_{\delta,\eta}(T,x)\zeta\xi(T,x)\,\mathrm{d}x\\ &- \int_\Omega u_0^{\mu,\delta,\epsilon}\zeta\xi(0,x)\,\mathrm{d}x - \int_Q \{\boldsymbol{\chi}(t,x,u_{\delta,\eta})-\boldsymbol{\chi}(t,x,0)\}.\nabla(\zeta\xi)\,\mathrm{d}x\mathrm{d}t\\ &+ \int_Q \big(Div_x\boldsymbol{\chi}(t,x,0)+\psi(t,x,u_{\delta,\eta})\big)\zeta\xi\,\mathrm{d}x\mathrm{d}t + \int_Q \nabla\phi_\delta(u_{\delta,\eta}).\nabla(\zeta\xi)\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

From a priori estimates of $u_{\delta,\eta}$ and $u_0^{\mu,\delta,\epsilon}$, we deduce the existence of a constant C independent from any parameter such that

$$|\langle\kappa_{\delta,\eta},\xi\rangle| \leq C\|\xi\|_\infty.$$

Now we are in the mathematical framework exposed in [5]: $(\kappa_{\delta,\eta})_{\eta>0}$ is a bounded sequence in $\mathcal{C}'(\overline{Q})$ used with the weak-$*$ topology. The latter and the previous inequality provide a bound for the limit in $\mathcal{C}'(\overline{Q})$ at each step when η and δ tend to 0^+. Besides this, the convergence properties of $(u_{\eta,\delta})_{\eta>0,\delta>0}$ permit one to pass to the limits in the right-hand side of (12). Consequently, there exists κ in $\mathcal{C}'(\overline{Q})$ such that $|\langle\kappa,\xi\rangle| \leq C\|\xi\|_\infty$ and

$$\forall\xi\in\mathcal{C}^\infty(\overline{Q}),\ \int_Q U_k\zeta.\overline{\nabla}\xi\,\mathrm{d}x\mathrm{d}t = \langle\kappa,\xi\rangle + I + sgn(k)J(\zeta\xi) - \int_\Sigma \mathbf{F}(0,k).\nu\zeta\xi\mathrm{d}\mathcal{H}^p. \tag{13}$$

where I is an integral bounded by $C\|\xi\|_\infty$. In (13) for $k=0$ and ξ in $\mathcal{D}(Q)$, we deduce that $Div_{(t,x)}(U_0\zeta)$ belongs to $\mathcal{M}_b(Q)$. For $k>0$, the positiveness of u ensures that for any ξ in $\mathcal{D}(Q)$,

$$\begin{aligned}J(\zeta\xi) = -\int_Q \big(&U_0\zeta.\overline{\nabla}\xi - \xi(Div_x\boldsymbol{\chi}(t,x,0)\zeta + \psi(t,x,u)\zeta\\ &+\{\boldsymbol{\chi}(t,x,u)-\boldsymbol{\chi}(t,x,0)-\nabla\phi(u)\}.\nabla\zeta)\big)\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

As a consequence $|J(\zeta\xi)| \leq C\|\xi\|_\infty$ and $U_k\zeta$ is in $\mathcal{DM}_2(Q)$, for any k in $\mathbb{R}^+$. The other statements of Theorem 3 are detailed in [8] and are developed directly from (10) and (11) with typical arguments for obtaining (6) and those exposed in [5] for (7). □

3.3.1. Statement for the initial data in $L^\infty(\Omega)$

We first observe that, the parameter ϵ being fixed, $(u_0^{\mu,\epsilon})_{\mu>0}$ remains in a bounded set of $BV(\Omega)$. The compact embedding of the latter space $L^1(\Omega)$ ensures that, up to a subsequence when μ goes to 0^+,

$(u_0^{\mu,\epsilon})_{\mu>0}$ goes to u_0^ϵ in $L^q(\Omega)$, for any finite q. On the other hand, by construction, $(u_0^\epsilon)_{\epsilon>0}$ goes to u_0 in $L^q(\Omega)$, $1 \le q < +\infty$. Thus by using a diagonal extraction process, we construct a sequence $(u_0^\omega)_{\omega>0}$ extracted from $(u_0^{\mu,\epsilon})_{\mu>0,\epsilon>0}$ such that $\lim_{\omega\to 0^+} u_0^\omega = u_0$ in $L^q(\Omega)$, $1 \le q < +\infty$, and a.e. on Ω.

Now we consider u_ω, the weak entropy solution to (1) and (2) associated with the initial data u_0^ω thanks to Theorem 3. If we refer to ω-uniform estimates developed in Theorem 2 we have:

Proposition 1. *There exists a positive constant C, independent from ω, such that*

$$\forall t \in [0,T],\ |u_\omega(t,.)| \le M(t) \text{ a.e. in } \Omega,\ \|\partial_t u_\omega\|_{L^2(0,T;H^{-1}(\Omega))} + \|\phi(u_\omega)\|_{L^2(0,T;H_0^1(\Omega))} \le C.$$

Besides the uniqueness, Theorem 1 warrants

Proposition 2. *If u_{ω_1} and u_{ω_2} are weak entropy solutions to (1) and (2) related to $u_0^{\omega_1}$ and $u_0^{\omega_2}$, then*

$$\forall t \in [0,T],\ \|u_{\omega_1}(t,.) - u_{\omega_2}(t,.)\|_{L^1(\Omega)} \le e^{M_\psi t}\|u_0^{\omega_1} - u_0^{\omega_2}\|_{L^1(\Omega)}.$$

Let us remark that the $L^1(Q)$-estimates in Theorem 2 are not ω-uniform since $\|\nabla u_0^{\mu,\delta,\epsilon}\|_{L^1(\Omega)^p}$ and $\|\Delta\phi_\delta(u_0^{\mu,\delta,\epsilon})\|_{L^1(\Omega)}$ depend on ϵ (through $\|\nabla u_0^\epsilon\|_{L^1(\Omega)^p}$) and $\frac{1}{\mu}$.

So $(u_\omega)_{\omega>0}$ is a Cauchy sequence in $\mathcal{C}([0,T]; L^1(\Omega))$ and up to a subsequence, Convergence (12), (13) also holds for $(u_\omega)_{\omega>0}$. By starting from (5)–(7) for u_ω and taking the ω-limit, we prove:

Theorem 4. *Let u_0 be in $L^\infty(\Omega)$ with $u_0 \ge 0$ a.e. in Ω. The degenerate parabolic–hyperbolic obstacle problem (1)–(3) admits at least a weak entropy solution in $\mathcal{C}([0,T]; L^q(\Omega))$ for any finite q.*

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