CONES RELATED TO THE LEFSCHETZ PROPERTIES

ROBERTA DI GENNARO, GIOVANNA ILARDI AND JEAN VALLÈS

Abstract. In [17] the authors highlight the link between rational varieties satisfying a Laplace equation and artinian ideals that fail the Weak Lefschetz property. Continuing their work we extend this link to the more general situation of artinian ideals failing the Strong Lefschetz Property. We characterize the failure of SLP (that includes WLP) by the existence of special singular hypersurfaces (cones for WLP). This characterization allows us to solve three problems posed in [18] and to give new examples of ideals failing the SLP. Finally, line arrangements are related to artinian ideals and the unstability of the associated derivation bundle is linked to the failure of SLP. Moreover we reformulate the so-called Terao’s conjecture for free line arrangements in terms of artinian ideals failing the SLP.

December 19, 2012

1. Introduction

The tangent space $T_aX$ to an integral projective variety $X$ of dimension $n$ in a smooth point has always dimension $n$. It is no longer true for the osculating spaces. For instance, as it was pointed out by Togliatti in [26], the osculating space $T^2_aX$, in a general point $a$, of the rational surface $X$ defined by

$$\mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}^5, (x, y, z) \mapsto (xz^2, yz^2, x^2z, y^2z, xy^2, x^2y),$$

has projective dimension 4 instead of 5. Indeed there is a non trivial linear relation between the partial derivatives of order 2 of $\phi$ in $a$ that define $T^2_aX$. This relation is usually called a Laplace equation of order 2. More generally, we will say that $X$ satisfies a Laplace equation of order $k$ when its $k$-th osculating space $T^k_aX$ in a general point $a \in X$ has dimension less than the expected one.

The study of the surfaces satisfying a Laplace equation was developed in the last century by Togliatti [26] and Terracini [25]. Togliatti [26] gave a complete classification of the rational surfaces embedded by linear systems of plane cubics and satisfying a Laplace equation of order two.

In the dense paper [21], Perkinson gives a complete classification of smooth toric surfaces (Theorem 3.2) and threefolds (Theorem 3.5) embedded by a monomial linear system and satisfying a Laplace equation of any order.

Very recently Miro-Roig, Mezzetti and Ottaviani [17] establish a nice link between rational varieties (i.e. projections of Veronese varieties) satisfying a Laplace equation and artinian graded rings $A = \oplus_{0 \leq i \leq s} A_i$ such that the multiplication by a general linear form has not maximal rank in a degree $i$. On the contrary, when the rank of the multiplication map is maximal in any degree, the ring is said to have the Weak Lefschetz
property (briefly WLP). This name comes from a Stanley’s preliminary work: the Hard Lefschetz theorem is used to prove that the ring \( C[x_0, \ldots, x_n] \) has the WLP [24, Theorem 2.4]. From this example one can ask if the artinian complete intersection rings have the WLP. Actually, \( C[x,y,z] \) has the WLP (first proved in [12] and then also in [3]) but it is still not known for more than three variables. Many other questions derive from this first example.

The same type of problems arises when we consider the multiplication by powers \( L^k \) (\( k \geq 1 \)) of a general linear form \( L \). Indeed, if the rank of the multiplication map by \( L^k \) is maximal for any \( k \) and any degree, the ring is said to have the Strong Lefschetz property (briefly SLP). For more details about known results and some open problems we refer to [18].

As in [12], many papers about the Lefschetz properties involve the syzygy bundle \( K \). It is associated to an artinian ideal \( I = (F_1, \ldots, F_r) \) in the following way:

\[
0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^n}^r (F_1, \ldots, F_r) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow 0.
\]

Indeed, in [3, prop. 2.1] Brenner and Kaid proved that the graded piece of degree \( d + i \) of the artinian ring \( A = \mathbb{C}[x_0, \ldots, x_n]/(F_0, \ldots, F_r) \) is \( H^1(K(i)) \). In [17, thm. 3.2] the authors characterize the failure of WLP (in degree \( d - 1 \), i.e. for the map \( A_{d-1} \rightarrow A_d \)) when \( r \leq h^0(\mathcal{O}_L(d)) \) by the non injectivity of the restricted map

\[
H^0(\mathcal{O}_L)^r (F_1, \ldots, F_r) \longrightarrow H^0(\mathcal{O}_L(d)),
\]

on a general hyperplane \( L \).

Let us say, in a few words, what we are doing in this paper and how it is organized. First of all we recall some definitions, basic facts and we propose a conjecture (Section 3). In Section 4 we extend to the SLP the previous characterization of failure of WLP. Then we translate the failure of WLP and SLP in terms of existence of special singular hypersurfaces (Section 5). It allows us to give an answer to three unsolved questions in [18]. In Section 6 we construct examples of artinian rings failing the WLP and the SLP by producing the appropriate singular hypersurfaces. In the last section we relate the problem of SLP at the range 2 to the topic of line arrangements (Section 7).

Let us now give more details about the different sections of this paper. In Section 4 more precisely in Theorem 4.1 we characterize the failure of SLP by the non maximality of the induced map on sections

\[
H^0(\mathcal{O}_L)^r (F_1, \ldots, F_r) \longrightarrow H^0(\mathcal{O}_L(i + d)).
\]

The geometric consequences of this link are explained in Section 5 (see Theorem 5.1). The non injectivity is translated in terms of the number of Laplace equations and the non surjectivity is related, via apolarity, to the existence of special singular hypersurfaces. Then we give Propositions 5.3, 5.4 and 5.5 that solve three problems posed in [18, Problem 5.4 and Conjecture 5.13].

In Section 6 we produce many examples of ideals (monomial and non monomial) that fail the WLP and SLP. The failure of WLP is studied for monomial ideals generated in degree 4 on \( \mathbb{P}^2 \) (Theorem 6.1), in degree 5 on \( \mathbb{P}^2 \) (Proposition 6.2), in degree 4 on \( \mathbb{P}^3 \) (Proposition 6.5); the failure of SLP is studied for monomial ideals generated in degree 4 (Proposition 6.6); finally, we propose a method to produce non monomial ideals that fail the SLP at any range (Proposition 6.7).

In the last section Lefschetz properties and line arrangements are linked. The theory of line arrangements, more generally of hyperplane arrangements, is an old and deep subject that concerns combinatorics, topology and algebraic geometry. One can say...
that it began with Jakob Steiner (in the first volume of Crelles’s journal, 1826) that
determined in how many regions a real plane is divided by a finite number of lines. It is
relevant also with the amazing Sylvester-Gallai’s problem. Hyperplane arrangements
come back in a modern presentation in the Arnold’s fundamental work [4] on the
cohomology ring of $\mathbb{P}^n \setminus D$ (where $D$ is the union of the hyperplanes of the arrangement).
For a large part of mathematicians, working on arrangements, it culminates today
with the Terao’s conjecture (see the last section of this paper or directly [20]). This
conjecture concerns particularly the derivation sheaf (also called logarithmic sheaf)
associated to the arrangement. In this paper we recall the conjecture. In Proposition
7.2 we prove that the failure of the SLP at the range 2 of some ideals is equivalent to
the unstability of the associated derivation sheaves. Thanks to the important literature
on arrangements, we find artinian ideals that fail the SLP. For instance the Coxeter
arrangement, called B3, gives an original ideal that fails the SLP at the range 2 in a
non trivial way (see Proposition 7.3).

We finish by a reformulation of Terao’s conjecture in terms of SLP.

2. Notations

The ground field is $\mathbb{C}$.
The dual $\text{Hom}_\mathbb{C}(V, \mathbb{C})$ of a vector space $V$ is denoted by $V^*$.
The dimension of the vector space $H^0(\mathcal{O}_{\mathbb{P}^n}(t))$ is denoted by $r_t$ where $n$ is clearly known
in the context.
The vector space generated by the set $E \subset \mathbb{C}^t$ is $\langle E \rangle$.
The join variety of $s$ projective varieties $X_i \subset \mathbb{P}^n$ is denoted by $\text{Join}(X_1, \cdots, X_s)$
(see [11] for the definition of join variety).
The fundamental points $(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,0,1)$ in $\mathbb{P}^n$ are denoted
by $P_0, P_1, \ldots, P_n$.
We often write in the same way a projective hyperplane and the linear form defining
it; we use in general the notation $L_i$ on $\mathbb{P}^n$ and the notation $l_i$ on $\mathbb{P}^2$ for hyperplanes.
The ideal sheaf of a point $P$ is $\mathcal{I}_P$.

3. Lefschetz properties

Let $R = \mathbb{C}[x_0, \ldots, x_n]$ be the graded polynomial ring in $n+1$ variables over $\mathbb{C}$. The
dimension of the vector space $R_i$ is $r_i$.

Let $A = R/I = \sum_{i=0}^{m} A_i$
be a graded artinian algebra. Note that $A$ is finite dimensional over $\mathbb{C}$.

Definition 3.1. Let $L$ be a general linear form. The artinian algebra $A$ (or the artinian
ideal $I$) has the Weak Lefschetz Property (WLP) if the homomorphism induced by the
multiplication by $L$,

$\times L : A_i \to A_{i+1},$

has maximal rank (i.e. is injective or surjective) for all $i$. The artinian algebra $A$ (or
the artinian ideal $I$) has the Strong Lefschetz Property (SLP) if,

$\times L^k : A_i \to A_{i+k},$

has maximal rank (i.e. is injective or surjective) for all $i$ and $k$.

Remarks. • It is clear that the SLP for $k = 1$ corresponds to the WLP.
We will often be interested in artinian rings $A$ that fail the SLP (or WLP), i.e. when for a general linear form $L$ there exist $i$ and $k$ such that the multiplication map

$$\times^k : A_i \to A_{i+k},$$

has not maximal rank. In that case we will say that $A$ (or $I$) fails the SLP at range $k$ and degree $i$. When $k = 1$ we will say simply that $A$ fails the WLP in degree $i$.

One of the main examples comes from the Togliatti’s result (see for instance [3], Example 3.1): the ideal $I = (x^3, y^3, z^3, xyz)$ fails the WLP in degree 2. There are many ways to prove it. One of them comes from the polarity on the rational normal cubic curve. It leads to a generalization that gives one of the few known non toric examples.

**Proposition 3.2** ([27], Thm. 3.1). Let $n \geq 1$ be an integer and $l_1, \ldots, l_{2n+1}$ be non concurrent linear forms on $\mathbb{P}^2$. Then the ideal

$$(l_1^{2n+1}, \ldots, l_{2n+1}^{2n+1}, \prod_{i=1}^{2n+1} l_i)$$

fails the WLP in degree $2n$.

Indeed on the general line $l$ the $2n + 2$ forms of degree $2n + 1$ become dependent thanks to the polarity on the rational normal cubic curve. We propose the following conjecture. For $n = 1$ it is again the Togliatti’s result.

**Conjecture.** Let $l_i$ be non concurrent linear forms on $\mathbb{P}^2$ and $f$ be a form of degree $2n + 1$ on $\mathbb{P}^2$. Then the ideal $(l_1^{2n+1}, \ldots, l_{2n+1}^{2n+1}, f)$ fails the WLP in degree $2n$ if and only if $f \in (l_1^{2n+1}, \ldots, l_{2n+1}^{2n+1}, \prod_{i=1}^{2n+1} l_i)$.

4. LEFSCHETZ PROPERTIES AND THE SYZYGY BUNDLE

In [17 Proposition 2.3], the failure of WLP in degree $d - 1$ is related to the restriction of the syzygy bundle to a general hyperplane. Here we extend this relationship to the SLP situation at any range and in many degrees.

**Theorem 4.1.** Let $I = (F_1, \ldots, F_r) \subset R$ be an artinian ideal generated by homogeneous forms of degree $d$ and $K$ the vector bundle defined by the exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}_d}^{\oplus 2n} \xrightarrow{\Phi_I} \mathcal{O}_{\mathbb{P}_d} \longrightarrow 0,$$

where $\Phi_I(a_1, \ldots, a_r) = a_1F_1 + \ldots + a_rF_r$. Let $i$ be an integer such that $h^0(K(i)) = 0$ and $k$ be an integer such that $k \geq 1$. Then $I$ fails the SLP at the range $k$ in degree $d + i - k$ if and only if the induced homomorphism on sections (denoted by $H^0(\Phi_{I,L^k})$)

$$H^0(\mathcal{O}_{\mathbb{P}_d}(i)) \xrightarrow{H^0(\Phi_{I,L^k})} H^0(\mathcal{O}_{\mathbb{P}_d}(i + d))$$

has not maximal rank for a general linear form $L$.

**Remark.** The theorem is not true if $h^0(K(i)) \neq 0$ i.e. if there exists a syzygy of degree $i$ among $F_1, \ldots, F_r$. In [17] the authors consider the injectivity of the map $H^0(\Phi_{I,L^k})$ for $i = 0$ and for $r \leq h^0(\mathcal{O}_L(d))$. In that case, since the forms $F_j$ are the generators of $I$, we have of course $h^0(K) = 0$. 
Proof. In [3, Proposition 2.1] the authors proved that \( A_{d+i} = H^1(K(i)) \) for any \( i \in \mathbb{Z} \).

Let us consider the canonical exact sequence

\[
0 \longrightarrow K(i - k) \xrightarrow{\times L^k} K(i) \longrightarrow K \otimes \mathcal{O}_{L^k}(i) \longrightarrow 0.
\]

We obtain a long exact sequence of cohomology

\[
0 \rightarrow H^0(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow A_{d+i-k} \xrightarrow{\times L^k} A_{d+i} \rightarrow H^1(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow H^2(K(i - k)) \rightarrow 0.
\]

Let us assume first that \( n > 2 \). Then we have always \( h^2(K(i - k)) = 0 \) and it gives a shorter exact sequence:

\[
0 \longrightarrow H^0(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow A_{d+i-k} \xrightarrow{\times L^k} A_{d+i} \rightarrow H^1(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow 0.
\]

Moreover, since \( n > 2 \), we have also \( h^1(\mathcal{O}_{L^k}(i)) = 0 \). Then by tensoring the exact sequence defining the syzygy bundle \( K \) by \( \mathcal{O}_{L^k}(i) \) and taking the long cohomology exact sequence, we find:

\[
0 \longrightarrow H^0(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow H^0(\mathcal{O}_{L^k}(i)) \xrightarrow{H^0(\Phi_{l,L^k})} H^0(\mathcal{O}_{L^k}(i+d)) \rightarrow H^1(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow 0.
\]

Since the kernel and cokernel of both maps, \( H^0(\Phi_{l,L^k}) \) and \( \times L^k \) are the same, the theorem is proved for \( n > 2 \).

If \( n = 2 \), let us introduce the number \( t = h^2(K(i - k)) \). This number is equal to \( t = \text{sup}(0, N(r, i, k)) \) and we have a long exact sequence:

\[
0 \longrightarrow H^0(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow A_{d+i-k} \xrightarrow{\times L^k} A_{d+i} \rightarrow H^1(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow \mathbb{C}^l \rightarrow 0.
\]

Let us consider now the long exact sequence:

\[
0 \longrightarrow H^0(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow H^0(\mathcal{O}_{L^k}(i)) \xrightarrow{H^0(\Phi_{l,L^k})} H^0(\mathcal{O}_{L^k}(i+d)) \longrightarrow H^1(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow H^1(\mathcal{O}_{L^k}(i+d)) \rightarrow 0.
\]

Since \( h^1(\mathcal{O}_{L^k}(i)) = h^2(\mathcal{O}_{L^k}(i) - k) = r_{k-i-3} \) (and \( h^1(\mathcal{O}_{L^k}(i+d)) = h^2(\mathcal{O}_{L^k}(i+d-k)) = r_{k-i-d-3} \)), it remains a shorter exact sequence

\[
0 \longrightarrow H^0(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow H^0(\mathcal{O}_{L^k}(i)) \xrightarrow{H^0(\Phi_{l,L^k})} H^0(\mathcal{O}_{L^k}(i+d)) \rightarrow H^1(K \otimes \mathcal{O}_{L^k}(i)) \rightarrow \mathbb{C}^l \rightarrow 0.
\]

As before, since the kernel and cokernel of both maps are the same, the theorem is proved.

Let us introduce the numbers \( N(r, i, k, d) := r(r_i - r_{i-k}) - (r_{d+i} - r_{d+i-k}) \),

\[
N^+ = \text{sup}(0, N(r, i, k, d)) \text{ and } N^- = \text{sup}(0, -N(r, i, k, d)).
\]

The following corollary is a didactic reformulation of the theorem above.

**Corollary 4.2.** Assume that there is no syzygy of degree \( i \) among the \( F_j \)'s. Then \( I \) fails the SLP at the range \( k \) in degree \( d + i - k \) if and only if one of the two following equivalent conditions occurs:

- \( h^0(K \otimes \mathcal{O}_{L^k}(i)) = \ker(H^0(\Phi_{l,L^k})) > N^+ \),
- \( \text{coker}(H^0(\Phi_{l,L^k})) > N^- \).

In the next section we translate this corollary in geometric terms.
5. SYZYGY BUNDLE AND VERONESE VARIETY

We recall that the $k$-th osculating space to a $n$-dimensional complex projective variety $X \subset \mathbb{P}^N$ at $P$ is the subspace of $\mathbb{P}^N$ spanned by $P$ and by all the derivative points of degree less than or equal to $k$ of a local parametrization of $X$, evaluated at $P$. A $n$-dimensional variety $X \subset \mathbb{P}^N$ whose $k$-th osculating space at a general point has dimension $\inf((n+k) - 1, N) - \delta$ is said to satisfy $\delta$ independent Laplace equations of order $k$. We will say, for shortness, that the number of Laplace equations is $\delta$.

Let us briefly explain now the link with projections of $v_d(\mathbb{P}^n)$.

Let $R_1$ be a complex vector space of linear forms of dimension $n + 1$ such that $H^0\mathcal{O}_{\mathbb{P}^n}(1) = R_1$. We consider the Veronese embedding:

$$v_d : \mathbb{P}(R_1^*) \hookrightarrow \mathbb{P}(R_d^*) \text{ where } [L] \mapsto [L^d].$$

The image $v_d(\mathbb{P}^n)$ is called Veronese $n$-fold of order $d$. At the point $[L^d] \in v_d(\mathbb{P}^n)$, the $s$-th osculating space, $1 \leq s \leq d - 1$, denoted by $T_{[L^d]}^s v_d(\mathbb{P}^n)$ and defined by the partial derivatives of order $s$, is the space of degree $d$ forms possessing a factorization $L^{d-s} G$ where $G$ is a form of degree $s$. It is identified with $\mathbb{P}(R_1^s)$.

Let us think about the projective duality in terms of derivations (it is in fact the so-called apolarity, see [6]). A canonical basis of $R_1^s$ is given by the $r_d$ derivations:

$$\frac{\partial^d}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}} \text{ with } i_1 + \ldots + i_n = d.$$

Let $I = (F_1, \ldots, F_r) \subset R$ an ideal generated by $r$ forms of degree $d$. Note that $F_i$ are points in $\mathbb{P}(R_1^s)$. We denote by $I_d$ the vector subspace of $R_d$ generated by the $F_i$’s.

Let us introduce the orthogonal vector space to $I_d$:

$$I_d^\perp = \{ \delta \in R_d^* | \delta(F) = 0, \forall F \in I_d \}.$$

It gives an exact sequence of vector spaces

$$0 \longrightarrow I_d^\perp \longrightarrow R_d^* \longrightarrow I_d^* \longrightarrow 0$$

and the corresponding projection map $\mathbb{P}(R_d^*) \setminus \mathbb{P}(I_d^*) \overset{\pi}{\longrightarrow} \mathbb{P}(I_d^\perp)$.

Of course one can identify $R_d/I_d \simeq (I_d^\perp)^*$ and write the decomposition $R_d = I_d \oplus (I_d^\perp)^*$.

**Remark.** In the following two situations, the vector space $(I_d^\perp)^*$ is easy to describe:

1. When $I_d$ is generated by $r$ monomials of degree $d$, $(I_d^\perp)^*$ is generated by the remaining $r_d - r$ monomials.

2. When $I_d = (L_1^d, \ldots, L_r^d)$ where $[L_i] \in \mathbb{P}(R_d^*)$, $(I_d^\perp)^*$ is generated by degree $d$ polynomials that vanish at the points $[L_i^d] \in \mathbb{P}(R_1)$.

It is well known that the tangent spaces to the Veronese varieties can be interpreted as singular hypersurfaces. More precisely a hyperplane containing the tangent space $T_{[L^d]} v_d(\mathbb{P}^n)$ corresponds to the dual space $\mathbb{P}^{n^\vee}$ to a hypersurface of degree $d$ that is singular at the point $[L^d]$. More generally a hyperplane containing the $s$-th ($s \leq 1$) osculating space $T_{[L^d]}^s v_d(\mathbb{P}^n)$ corresponds to a hypersurface of degree $d$ and multiplicity $(s + 1)$ at the point $[L^d]$ (see for instance [2]).

Thus the dual variety of $v_d(\mathbb{P}^n)$ is the discriminant variety that parametrizes the singular hypersurfaces of degree $d$ when the $s$-th osculating variety of $v_d(\mathbb{P}^n)$ parametrizes the hypersurfaces of degree $d$ with a point of multiplicity $s + 1$.

We propose now an extended version of the “main” theorem of [17] (to be precise Theorem 3.2).
Theorem 5.1. Let $I = (F_1, \ldots, F_r) \subset R$ be an artinian ideal generated by $r$ homogeneous polynomials of degree $d$. Let $i, k, \delta$ be integers such that $i \geq 0$, $k \geq 1$. Assume that there is no syzygy of degree $i$ among the $F_j$'s. The following conditions are equivalent:

1. The ideal $I$ fails the SLP at the range $k$ in degree $d + i - k$.
2. There exist $N^+ + \delta$, with $\delta \geq 1$, independent vectors $(G_{1j}, \ldots, G_{rj})_{j=1, \ldots, N^+ + \delta} \in R^{\mathbb{P}^n}$ and $N^+ + \delta$ forms $G_j \in R^{d+i-k}$ such that $G_{1j}F_1 + \ldots + G_{rj}F_r = L^kG_j$ for a general linear form of $L$ of $\mathbb{P}^n$.
3. The $n$-dimensional variety $\pi_I(v_{d+i}(\mathbb{P}^n))$ satisfies $\delta \geq 1$ Laplace equations of order $d + i - k$.
4. For any $L \in R_1$, dim$_C((I_{d+i})^\ast \cap H^0(I_{d+i}^{d+i-k+1}(d+i)) \geq N^+ + \delta$, with $\delta \geq 1$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is proved in Theorem 4.1.

Since $I$ is generated in degree $d$, the map $R_i \times I_d \rightarrow I_{d+i}$ is surjective and the relation $G_1F_1 + \ldots + G_rF_r = L^kG$ is equivalent to $\mathbb{P}(I_{d+i}^\ast) \cap T_{d+i}^{d+i-k}v_{d+i}(\mathbb{P}^n) \neq \emptyset$. More generally the number of independent relations $G_1F_1 + \ldots + G_rF_r = L^kG_j$ is the dimension of the kernel of the map $H^0(\Phi_{I, L^k})$ i.e. the dimension of $H^0(K \otimes \mathcal{O}_{L^k})$; this number of independent relations, written in a geometric way, is

$$\delta = \dim[\mathbb{P}(I_{d+i}^\ast) \cap T_{d+i}^{d+i-k}v_{d+i}(\mathbb{P}^n)] + 1,$$

where the projective dimension is $-1$ if the intersection is empty. This number is the number of Laplace equations. Indeed, the dimension of the $(d + i - k)$-th osculating space to $\pi_I(v_{d+i}(\mathbb{P}^n))$ is $r_{d+i-k} - \delta$ since the $(d + i - k)$-th osculating space to $v_{d+i}(\mathbb{P}^n)$ meets the center of projection along a $\mathbb{P}^{d-1}$. In other words, the $n$-dimensional variety $\pi_I(v_{d+i}(\mathbb{P}^n))$ satisfies $\delta$ Laplace equations and (3) is equivalent to (2).

The image by $\pi_I$ of the $(d + i - k)$-th osculating space to the Veronese $v_{d+i}(\mathbb{P}^n)$ in a general point has codimension $h^0(K \otimes \mathcal{O}_{L^{d+i}}) - N^+ \geq \mathbb{P}(I_{d+i}^\ast)$. The codimension corresponds to the number of hyperplanes in $\mathbb{P}(I_{d+i}^\ast)$ containing the osculating space to $\pi_I(v_{d+i}(\mathbb{P}^n))$. These hyperplanes are images by $\pi_I$ of hyperplanes in $\mathbb{P}(R_{d+i})$ containing $\mathbb{P}(I_{d+i}^\ast)$ and the $(d + i - k)$-th osculating plane to $v_{d+i}(\mathbb{P}^n)$ at the point $[L^{d+i}]$. In the dual setting it means that these hyperplanes are forms of degree $d + i$ in $(I_{d+i})^\ast$ with multiplicity $(d + i - k + 1)$ at $[L^d]$. It proves that (3) is equivalent to (4).

To summarize, the number of Laplace equation is $h^0(K \otimes \mathcal{O}_{L^{d+i}}) - N^+$ and $\text{coker}(H^0(\Phi_{I, L^{d+i}}) \cong (I_{d+i})^\ast \cap H^0(T_{d+i}^{d+i-k+1}(d+i))$. \hfill $\Box$

Remarks.

- Let $I = (L_1^d, \ldots, L_r^d)$ where $L_1, \ldots, L_r$ be general linear forms. The vector space $(I_{d+i})^\ast$, where $I_{d+i} = L_1R_i + \ldots + L_rR_i$, is the vector space of the forms of degree $d + i$ vanishing in $r$ points $[L_j^d]$ with multiplicity $(i + 1)$. In other words $f \in \cap_{j=1}^r H^0(T_{L_j^d}^{d+i}(d+i))$ (see Corollary 3). Geometrically it can be described as $\mathbb{P}(I_{d+i}^\ast) = \text{Join}(T_{d+i}^{d+i}(v_{d+i}(\mathbb{P}^n))$).

- By the theorem above, when $N(r, i, d, k) \geq 0$, the ideal $I = (L_1^d, \ldots, L_r^d)$ fails the SLP at the range $k$ in degree $d - k + i$ if and only if the following intersection is not empty:

$$\text{Join}(T_{d+i}^{d+i-k}(v_{d+i}(\mathbb{P}^n)) \cap T_{d+i}^{d+i-k}(v_{d+i}(\mathbb{P}^n)) \cap T_{d+i}^{d+i-k}(v_{d+i}(\mathbb{P}^n)) \cap T_{d+i}^{d+i-k}(v_{d+i}(\mathbb{P}^n))$$

- On $\mathbb{P}^2$ a hypersurface of degree $d + i$ with a point of multiplicity $d + i$ is an union of lines. On $\mathbb{P}^n$, with $n > 2$, a hypersurface of degree $d + i$ with a point of multiplicity $d + i$ is more generally a cone over a hypersurface in the hyperplane at infinity.
• Here we focus the attention also on the number δ of Laplace equations satisfied by \( \pi_L(\mathcal{O}_{\mathbb{P}^d}^d) \). The geometric meaning of this number was highlighted by Terracini [25] for Laplace equations of order 2 and recently for any order by [8], where a classification of varieties satisfying “many” Laplace equations is given.

Thanks to this remark, in the three following propositions we can answer to a list of questions posed by Migliore and Nagel ( [18, Problem 5.4 and Conjecture 5.13]). When \( I = (L_1^d, \ldots, L_r^d) \) we verify now that the hypothesis in Theorem 5.1 on the syzygy is not restrictive.

**Lemma 5.2.** Let \( I \) be the ideal \((L_1^d, \ldots, L_r^d)\) where the \( L_j \) are linear forms and \( r < d \). Let \( K \) be its syzygy bundle. Then

\[
h^0(K(i)) = 0 \iff rr_i \leq r_{d+i}.
\]

**Proof.** One direction is obvious. Let us assume that \( rr_i \leq r_{d+i} \) and that there exists a relation

\[
G_1L_1^d + \ldots + G_rL_r^d = 0,
\]

with \( G_1, \ldots, G_r \) forms of \( R_i \). Both hypotheses imply that the projective space \( \operatorname{Join}(T_{L_1^d}^i \mathcal{O}_{\mathbb{P}^d}^d, \ldots, T_{L_r^d}^i \mathcal{O}_{\mathbb{P}^d}^d) \) has dimension strictly less than the expected one. Since the linear forms are general, it implies that the algebraic closure of \( \mathcal{V}(I) \) has not the expected dimension. It contradicts the lemma \[3.3 \text{ in [2].} \]

The following proposition is a generalization of the [13, Example 1.1].

**Proposition 5.3.** Let \( N \) be an integer such that \( N \geq 3 \). Then the ideal \((x_N^N, x_1^N, x_2^N, x_3^N, (x_0 + x_1 + x_2 + x_3)^N)\) fails the WLP in degree \( N \geq 2 \).

**Remark.** Of course it is equivalent to say that \((L_1^N, \ldots, L_5^N)\) fails the WLP in degree \( N = 2 \) for \( L_1, \ldots, L_5 \) general linear forms.

**Proof.** Let us consider the syzygy bundle associated to the linear system

\[
0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^3}^d \longrightarrow (x_0^N, x_1^N, x_2^N, x_3^N, (x_0 + x_1 + x_2 + x_3)^N) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(N) \longrightarrow 0.
\]

Since \( 5r_{N-2} < r_{2N-2} \) Lemma 5.2 implies \( h^0(K(N - 2)) = 0 \). Let \( L \) be a linear form. When \( N \geq 3 \) we have \( 5h^0(\mathcal{O}_L(N - 2)) \geq h^0(\mathcal{O}_L(2N - 2)) \). According to Theorem 5.1 the failure of WLP in degree \( N - 2 \) is equivalent to the existence of \( 1 \) of \( N - 1 \) in the points \( P_0, P_1, P_2, P_3, P \) and \( (1, 1, 1, 1) \) and multiplicity \( 2N - 2 \) at a moving point \( M \). The five concurrent lines in \( M \) passing through \( P_0, P_1, P_2, P_3, P \) belong to a quadric cone with equation \( \{ F = 0 \} \) (the cone over the conic at infinity through the five points). Since \( F^{N-1} \in H^0(\mathcal{O}_{\mathbb{P}^3}^d(2N - 2)) \) the hypersurface \( \{ F^{N-1} = 0 \} \) has the wanted properties.

The same idea gives a hint to answer the third question in the above list of problems (Problem 5.4). Indeed, in \( \mathbb{P}^n \) there is always a quadric through \( \frac{n(n+3)}{2} \) points in general position. Then given any general point \( M \in \mathbb{P}^{n+1} \), there is a quadric cone with a vertex at \( M \) and passing through \( \frac{n(n+3)}{2} \) fixed points in general position. Then we prove,

**Proposition 5.4.** In the following cases the ideal \((L_1^N, \ldots, L_{N(n+3)}^N)\) fails the WLP in degree \( N - 2 \):

- \( N = 3 \) and \( n \geq 2 \),
- \( N = 4 \) and \( 2 \leq n \leq 4 \),
• $N > 4$ and $2 \leq n \leq 3$.

**Proof.** Let us consider the syzygy bundle associated to the linear system

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}^{\frac{n(n+3)}{2}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(N) \longrightarrow 0.$$  

Let $L$ a linear form. Then the inequality $\frac{n(n+3)}{2} h^0(\mathcal{O}_L(N-2)) \geq h^0(\mathcal{O}_L(2N-2))$ is true if and only if $N$ and $n$ are one of the possibilities stated in the theorem. In all these cases we have $\frac{n(n+3)}{2} r_{N-2} \leq r_{2N-2}$, and by Lemma 5.2 $h^0(K(N-2)) = 0$.

According to Theorem 5.1 the failure of WLP is equivalent to the existence of a hypersurface with multiplicity $N - 1$ in the points $[L_y^7]$ and multiplicity $2N - 2$ at the moving point $M$. The lines through $M$ and $[L_y^7]$ belong to a quadratic cone with equation $\{F = 0\}$ (the cone over the quadric at infinity through the points). Since $F^{N-1} \in H^0(\mathcal{I}_M^{2N-2}(2N-2))$ the hypersurface $\{F^{N-1} = 0\}$ has the wanted properties. \(\square\)

The same idea gives the proof of the following result that was conjectured in [16].

**Proposition 5.5.** The ideal $I = (L_1^3, \ldots, L_8^3)$ fails the WLP in degree 3 where $L_1, \ldots, L_8$ be general linear forms on $\mathbb{P}^6$.

**Proof.** Since $8r_1 < r_4$ Lemma 5.2 implies $h^0(K(1)) = 0$. We have to prove that, on a general hyperplane $L$, the cokernel of $H^0(\mathcal{O}_L(1))^8 \longrightarrow H^0(\mathcal{O}_L(4))$ has dimension strictly greater than $h^0(\mathcal{O}_L(4)) - h^0(\mathcal{O}_L(1))^8 = 78$. The dimension of this cokernel is the dimension of the quartics with a quadruple point $[L'_y]$ and 8 double points. We consider on the hyperplane at infinity the vector space $V$ of quadrics through the images of the 8 points $[L_1'^7], \ldots, [L_8'^7]$. It has dimension 13. Let $Q_1, \ldots, Q_{13}$ be a basis of this space of quadrics. Then the vector space $\text{Sym}^2(V)$ of quartics generated by the products $Q_iQ_j$ has dimension 91 and all these quartics are singular in the 8 points. In $\mathbb{P}^6$ the independent quartic cones over these quartics belong to the cokernel. \(\square\)

The characterization of SLP in Theorem 5.1 allows us to give, in the next section, many examples of ideals failing the WLP or the SLP by producing ad-hoc singular hypersurfaces.

6. **Classes of ideals failing the WLP and SLP**

6.1. **Monomial ideals coming from singular hypersurfaces.** In their nice paper about osculating spaces of Veronese surfaces, Lanteri and Mallavibarena remark that the equation of the curve given by three concurrent lines depends only on six monomials instead of seven. More precisely let us consider a cubic with a triple point at $(a, b, c)$ passing through $P_0$, $P_1$ and $P_2$. Its equation is $(bz - cy)(az - cx)(ay - bx) = 0$ and it depends only on the monomials $x^2y, xy^2, x^2z, xz^2, y^2z, yz^2$. So there is a non zero form in

$$(I_3^*)^* = <x^2y, xy^2, x^2z, xz^2, y^2z, yz^2> \subset R_3$$

that is triple at a general point. By this way they explain the Togliatti surprising phenomena ([15], Theorem 4.1), [14] and [9]).

We apply this idea in our context. Recall that in the monomial case being artinian to the ideal $I$ means that it contains the forms $x_0^d, \ldots, x_n^d$. Let us consider the $(n + 1)$ fundamental points $P_0, P_1, \ldots, P_n$ and let us assume that the number $r$ of monomials generating $I$ is chosen such that $N(r, i, k, d) = 0$ for $i \geq 0, k \geq 1$ fixed integers. Then, as it is proved in Theorem 5.1, the ideal $I$ fails the SLP at the range $k$ in degree $d + i - k$ if and only if there exists at any point $M(a_0, \ldots, a_n)$ a hypersurface of degree
$d + i$ with multiplicity $d + i - k + 1$ at $M$ given by a form in $(I_{d+i})^* \simeq R_{d+i}/I_{d+i}$. We have to write this equation with a number of monomials as small as possible. Then the orthogonal space becomes bigger and we will cover all the possible choices.

First of all we describe exhaustively the monomial ideals $(x^4, y^4, z^4, f, g) \subset \mathbb{C}[x, y, z]$ of degree 4 that do not verify the WLP.

**Theorem 6.1.** Up to permutation of variables the monomial ideals generated by five quartic forms in $\mathbb{C}[x, y, z]$ that fail the WLP in degree 3 are the following

- $I_1 = (x^4, y^4, z^4, x^3z, x^3y)$,
- $I_2 = (x^4, y^4, z^4, x^2y^2, xyz^2)$.

**Remark.** Geometrically it is evident that the first ideal $(x^4, y^4, z^4, x^3z, x^3y)$ fails the WLP. Indeed under the Veronese map, a linear form $L$ becomes a rational normal curve of degree four that defines a projective space $\mathbb{P}^4$ and, modulo $L$, the restricted monomials $\bar{x}^4\bar{y}^4$ can be interpreted as points of this $\mathbb{P}^4$. Then the tangent line to the rational quartic curve at the point $[\bar{x}^4]$ contains the two points $[\bar{x}^3\bar{y}]$ and $[\bar{x}^3\bar{z}]$. This line meets the plane $\mathbb{P}(\bar{x}^4, \bar{y}^4, \bar{z}^4 >)$ in one point; it implies that

$$\dim_{\mathbb{C}} < \bar{x}^4, \bar{y}^4, \bar{z}^4, \bar{x}^3\bar{y}, \bar{x}^3\bar{z} > \leq 4.$$ 

For the second ideal, it is not evident to see that the line $\mathbb{P}(\bar{x}^2\bar{y}^2, \bar{x}\bar{y}\bar{z}^2 >)$ always (for any restriction) meets the plane $\mathbb{P}(\bar{x}^4, \bar{y}^4, \bar{z}^4 >)$.

**Proof.** Let us consider the points $P_0$ , $P_1$ and $P_2$ and the degree 4 curves with a quadruple point in $(a, b, c)$ passing through these three points. These curves are product of four lines:

$$f(x, y, z) = (ay - bx)(cx - az)(cy - bz)(\alpha(ay - bx) + \beta(cx - az)) = 0.$$ 

Developing $f$ explicitly in the coordinates $(x, y, z)$, we see that the forms $x^4, y^4, z^4$ are missing and that twelve monomials appear to write its equation. Since we want only ten monomials, we have to remove two. The following possibilities occur:

- $\alpha = 0$ (or equivalently by permutation of variables $[\beta = 0]$ or $[\alpha \neq 0, \beta \neq 0$ and $b\alpha = c\beta]$) then the remaining linear system is $(x^4, y^4, z^4, y^3z, x^2y^3)$. It corresponds to the first case i.e. to the ideal $I_1$.

- $\alpha \neq 0$ and $\beta \neq 0$ but $c\beta + b\alpha = 0$ (or equivalently by permutation of variables $[2b\alpha - c\beta = 0]$ or $[b\alpha - 2c\beta = 0]$) then the remaining linear system is $(x^4, y^4, z^4, x^2yz, y^2z^3)$. It corresponds to the second case i.e. to the ideal $I_2$. 

\[\square\]
Remark. The quartic curve with multiplicity four in $(a, b, c)$ consists, in the first case, of two lines and a double line that are concurrent; in the second case of four concurrent lines in harmonic division.

We do not apply the same technique to describe exhaustively the monomial ideals $(x^5, y^5, z^5, f, g, h) \subset \mathbb{C}[x, y, z]$ of degree 5 that do not verify the WLP because the computations become too tricky. But we can give some cases by geometric arguments.

**Proposition 6.2.** The following monomial ideals

- $(x^5, y^5, z^5, x^3y^2, x^3z^2, x^3yz)$,
- $(x^5, y^5, z^5, x^4z, x^4y, m)$, where $m$ is any monomial,
- $(x^5, y^5, z^5, x^3y^2, x^2y^3, x^2y^2z)$,

fail the WLP in degree 4.

**Proof.** Under the Veronese map, a linear form fails the WLP in degree 4.

In the same way the osculating plane at $\bar{\mathcal{P}}_i$ consists, in the first case, of two lines and a double line that are concurrent; in the second case of four concurrent lines in harmonic division.

We do not apply the same technique to describe exhaustively the monomial ideals $(x^5, y^5, z^5, f, g, h) \subset \mathbb{C}[x, y, z]$ of degree 5 that do not verify the WLP because the computations become too tricky. But we can give some cases by geometric arguments.

**Proposition 6.3.** The monomial ideals $I = (x^3, y^3, z^3, t^3, f_1, f_2, f_3, f_4, f_5, f_6)$ where the forms $f_i$ are chosen among one of the following sets of monomials:

- $\{x^2y, xy^2, x^2z, x^2t, y^2z, y^2t, z^2t, yz^2, yzt, xyz, xyt\}$, (Case (A1))
- $\{x^2y, y^2x, x^2z, y^2z, x^2t, y^2t, z^2t\}$, (Case (A2))
- $\{x^2y, xy^2, z^2t, yz^2, x^2t, y^2t, z^2t\}$, (Case (A3))
- $\{x^2z, y^2z, xyz, xyt, x^2y, xy^2, z^2t, z^2t\}$, (Case (A4))
- $\{x^2y, xy^2, x^2z, x^2t, x^2t, yz^2, x^2t, yzt, xyt, yzt\}$, (Case (B1))

fail the WLP in degree 2.

**Remark.** We do not know if under permutation of variables the description above is exhaustive or not. The singular cubic that we are considering here are union of concurrent planes and not all the cubic cones.

**Proof.** We look for a surface of degree 3 with multiplicity 3 at a general point $M(a, b, c, d)$ that passes through the points $P_0, P_1, P_2, P_3$ such that its equation depends only on the remaining monomials in $R_3/I_3$. Such a cubic surface is a cone over a cubic curve. Here, instead of a general cubic cone we consider only three concurrent planes. Since these 3 planes have to pass through $P_0, P_1, P_2$ and $P_3$ it remains only, after a simple verification, the following cases:
(A) Case (A1).

(B) Case (A2).

(C) Case (A3).

(D) Case (A4).

(E) Case (B1)

- (A1) The equation of the cubic is \((bx - ay)(dz - ct)^2 = 0\).
- (A2) The equation of the cubic is \((bx - ay)(dz - ct)(cx - az) = 0\).
- (A3) The equation of the cubic is \((bx - ay)(dz - ct)(bx + ay + udz + uct) = 0\) where at any point \((a, b, c, d)\) the function \(u(a, b, c, d)\) verifies \(ab + u(a, b, c, d)cd = 0\).
- (A4) The equation of the cubic is \((bx - ay)(dz - ct)(bdx + ady - 2abt) = 0\).
- (B1) The equation of the cubic is \((cy - bz)(dz - ct)(dy - bt) = 0\).

\[\square\]

If we want \(I_3\) to be of dimension \(r < 10\) (for instance \(r = 8\)) we need \(10 - r + 1\) independent cubics with a triple point. So, to get the failure of WLP, we need \(10 - r + 1\) independent cubics with a triple point. Let us recover with our method two linear systems of eight cubic forms (the complete classification is already done in [17, Theorem 4.10]) that fail the WLP in degree 2.

**Proposition 6.4.** The following monomial ideals

- \(I = (x^3, y^3, z^3, t^3, x^2y, xy^2, zt^2, z^2t)\),
- \(J = (x^3, y^3, z^3, t^3, xyz, xyt, xzt, yzt)\)

fail the WLP in degree 2.

**Remark.** The ideals \(I\) and \(J\) correspond respectively to the cases (3) and (1) in [17, Theorem 4.10].

**Proof.** Let us consider the following three forms defining singular cubics passing through the fundamental points and a general point \((a, b, c, d)\):

\((ct - dz)(at - dx)(ay - bx) = 0, (ct - dz)^2(ay - bx) = 0, (ct - dz)(ay - bx)^2 = 0\).
They are particular cases of type $(A)$ in the proof of Proposition 6.3. They are linearly independent and can be written with twelve monomials. Then it remains only 8 forms for $I_3$:

$$I = (x^3, y^3, z^3, t^3, x^2y, xy^2, z^2t, z^2t).$$

Let us consider the following three forms defining singular cubics passing through the basis points and the general point $(a, b, c, d)$:

$$(bz - cy)(az - cx)(ay - bx) = 0, (bx - ay)(at - dx)(dy - bt) = 0, (az - cx)(dx - at)(dz - ct) = 0.$$

They are cases of type $(B1)$ in the proof of Proposition 6.3. They are linearly independent and can be written with twelve monomials:

$$(x^2y, x^2z, xy^2, xz^2, y^2z, yz^2, t^2y, t^2z, ty^2, tz^2, t^2x, x^2t).$$

It remains only

$$J = (x^3, y^3, z^3, xyz, xyt, xzt, yzt).$$

Of course the same argument (concurrent planes or hyperplanes) can be used in degree or dimension bigger than 3. For instance let us give a set of monomial ideals in $\mathbb{C}[x, y, z, t]$, generated in degree 4 that fail the WLP.

**Proposition 6.5.** Let $f_1, \ldots, f_{11}$ be eleven monomials chosen among

$$x^3y, x^3z, x^3t, xy^3, xz^3, xt^3, y^3z, y^3t, yz^3, z^3t, zt^3, x^2y^2, y^2z^2, x^2z^2.$$

Then the ideal $I = (x^4, y^4, z^4, t^4, f_1, \ldots, f_{11})$ fails the WLP in degree 3.

**Proof.** At any point $M = (a, b, c, d)$ an equation of a surface of degree 4 with multiplicity 4 at $M$ that passes through the points $P_0, P_1, P_2, P_3$ is given by

$$f(x, y, z, t) = (ct - dz)(at - dx)(ay - bx)(bz - cy) = 0.$$

We conclude this section with an example that fails the SLP at the range 2.

**Proposition 6.6.** The ideal $I = (x^4, y^4, z^4, xy^3, xz^3, x^2yz, y^2z^2, y^3z, yz^3) \subset \mathbb{C}[x, y, z]$ fails the SLP at the range 2 in degree 2.

**Proof.** Let $P_0, P_1, P_2$ and $M(a, b, c)$ be four points. We consider the quartic curve consisting of the union of the four lines $(MP_0), (MP_1), (MP_2)$ and $(P_0P_1)$. It is a quartic passing through $P_0, P_1, P_2$ and triple at $M(a, b, c)$. It depends on the six monomials $x^3y, x^3z, x^2y^2, x^2z^2, x^2yz, x^2z^2$. Then it remains 9 = 15 − 6 monomials

$$I_4 = \langle x^4, y^4, z^4, xy^3, xz^3, x^2yz, y^2z^2, y^3z, yz^3 \rangle.$$

The associated syzygy bundle $K$ verifies $h^0(K \otimes \mathcal{O}_L) \neq 0$ for a general linear form $L$. It proves that $I = (x^4, y^4, z^4, xy^3, xz^3, x^2yz, y^2z^2, y^3z, yz^3)$ fails the SLP at the range 2 in degree 2.

### 6.2. Non monomial examples coming from singular hypersurfaces.

Let us study now the interesting case $I_4^d = \mathcal{H}^0(\mathcal{I}_Z(d))^*$ where $Z$ is a finite set of distinct points in $\mathbb{P}^{2N}$ of length $|Z|$ and $\mathcal{I}_Z$ its ideal sheaf. The set $Z$ corresponds by projective duality to a set of $|Z|$ distinct lines in $\mathbb{P}^2$ defined by linear forms $l_1, \ldots, l_{|Z|}$. We will now consider the ideal $I \subset R$ generated by $(l_1^d, \ldots, l_{|Z|}^d)$. We have $|Z| = \dim \mathcal{I}_d$. 

Proposition 6.7. Let \( k \geq 1 \), \( r = r_d - r_{d-k} \) and \( Z = \{ l_1', \ldots, l_r' \} \) a finite set of \( r \) distinct points in \( \mathbb{P}^{2r} \) where \( l_i \) are linear forms on \( \mathbb{P}^2 \). Assume that there exists a subset \( Z_1 \subset Z \), of length \( r - d + k - 1 \), contained in a curve \( \Gamma_1 \) of degree \( k - 1 \). Then the ideal \( I = (l_1', \ldots, l_r') \) fails the SLP at the range \( k \) in degree \( d - k \).

Proof. The union of \( \Gamma_1 \) and \( (d - k + 1) \) concurrent lines at a point \( P \) passing through the remaining points \( Z \setminus Z_1 \), is a non zero section of \( H^0(I_Z \otimes \mathcal{T}_P^{d-k+1}(d)) \). By Theorem 5.1 it proves that \( I \) fails the SLP at the range \( k \) in degree \( d - k \). \( \square \)

With this method it is always possible to find systems of any degree that fail the SLP by exhibiting a curve of degree \( d \) with multiplicity \( d - k + 1 \) at a general point \( P \). But one can find some set of points for which these special curves do not split as product of lines (see Proposition 7.3 in the next section).

7. SLP at the range 2 and line arrangements on \( \mathbb{P}^2 \)

A line arrangement is a collection of distinct lines in the projective plane. Arrangement of lines or more generally of hyperplanes is a famous and classical object related to combinatorics, commutative algebra, topology and geometry. It has been studied by many different authors for a very long time (see [4] or [20] for a good introduction to this topic).

Let us denote by \( \{ f = 0 \} \) the equation of the union of lines of the considered arrangement. Another classical object associated to the arrangement is the vector bundle \( D_0 \) defined as the kernel of the jacobian map:

\[
0 \longrightarrow D_0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}^3 \xrightarrow{(\partial f)} \mathcal{O}_{\mathbb{P}^2}(d - 1).
\]

The bundle \( D_0 \) is called derivation bundle (sometimes logarithmic bundle) of the line arrangement (see [22] and [23] for an introduction to derivation bundles).

One can think about the lines of the arrangement in \( \mathbb{P}^2 \) as a set of distinct points \( Z \) in \( \mathbb{P}^{2r} \). Then we will denote by \( D_0(Z) \) the associated derivation bundle.

The arrangement of lines is said free with exponents \( (a, b) \) when its derivation bundle splits on \( \mathbb{P}^2 \) as a sum of two line bundles, more precisely when

\[
D_0(Z) = \mathcal{O}_{\mathbb{P}^2}(-a) \oplus \mathcal{O}_{\mathbb{P}^2}(-b).
\]

The splitting of \( D_0(Z) \) over a line \( l \subset \mathbb{P}^2 \) is related to the existence of curves (with a given degree \( a + 1 \)) passing through \( Z \) that are multiple (with multiplicity \( a \)) at \( l' \in \mathbb{P}^{2r} \). More precisely,

Lemma 7.1. ([24, Proposition 2.1]) Let \( Z \subset \mathbb{P}^{2r} \) be a set of \( a + b + 1 \) distinct points with \( 1 \leq a \leq b \) and \( l \) be a general line in \( \mathbb{P}^2 \). Then the following conditions are equivalent:

1. \( D_0(Z) \otimes \mathcal{O}_l = \mathcal{O}_l(-a) \oplus \mathcal{O}_l(-b) \).
2. \( h^0((\mathcal{J}_Z \otimes \mathcal{J}_{l'})^a(a + 1)) \neq 0 \) and \( h^0((\mathcal{J}_Z \otimes \mathcal{J}_{l'}^{a-1})(a)) = 0 \).

In our context it implies the following characterization of unstability. We recall that a rank two vector bundle \( E \) on \( \mathbb{P}^n \), \( n \geq 2 \) is unstable if and only if its splitting \( E_l = \mathcal{O}_l(a) \oplus \mathcal{O}_l(b) \) on a general line \( l \) verifies \( |a - b| \geq 2 \). This characterization is a consequence of the Grauert-Mülich theorem, see [19].

Proposition 7.2. Let \( I \subset R = \mathbb{C}[x, y, z] \) be an artinian ideal generated by \( 2d + 1 \) polynomials \( l_1^d, \ldots, l_{2d+1}^d \) where \( l_i \) are distinct linear forms in \( \mathbb{P}^2 \). Let \( Z = \{ l_1', \ldots, l_{2d+1}' \} \) be the corresponding set of points in \( \mathbb{P}^{2r} \). Then the following conditions are equivalent:

1. The ideal \( I \) fails the SLP at the range 2 in degree \( d - 2 \).
(2) The derivation bundle $\mathcal{D}_0(Z)$ is unstable.

Proof. The failure of SLP at the range 2 in degree $d - 2$ is equivalent to the existence at a general point $l^\vee$ of a curve of degree $d$ with multiplicity $d - 1$ at $l^\vee$ belonging to $I_d^\perp = H^0(I_Z(d))$. By the lemma 7.1 it is equivalent to the following splitting

$$\mathcal{D}_0(Z) \otimes \mathcal{O}_l = \mathcal{O}_l(d - s) \oplus \mathcal{O}_l(d + s)$$

with $s > 0$, on a general line $l$. In other words the failure of SLP is equivalent to have a non balanced decomposition and according to Grauert-Mülich theorem it is equivalent to unstability. □

Let us give now an ideal generated by non monomials quartic forms that fails the SLP at the range 2. It comes from a line arrangement, called B3 arrangement (see [20, Pages 13, 25 and 287]), such that the associated derivation bundle is unstable (in fact even decomposed). The existence of a quartic curve with a general triple point is the key argument. But contrary to the previous examples, this quartic is irreducible and consequently not obtainable by Proposition 6.7.

Proposition 7.3. The ideal

$$I = (x^4, y^4, z^4, (x + y)^4, (x - y)^4, (x + z)^4, (y + z)^4, (y - z)^4, (y + z)^4, (y - z)^4) \subset \mathbb{C}[x, y, z]$$

fails the SLP at the range 2 and degree 2.

Proof. Consider the set $Z$ of the nine dual points of the linear forms $x, y, z, x + y, x - y, x + z, x - z, y + z, y - z$. Let $I$ be the artinian ideal $(x^4, y^4, z^4, (x + y)^4, (x - y)^4, (x + z)^4, (y + z)^4, (y - z)^4)$ and $K$ its syzygy bundle. The derivation bundle of the arrangement is $\mathcal{D}_0(Z) = \mathcal{O}_{\mathbb{P}^2}(-3) \oplus \mathcal{O}_{\mathbb{P}^2}(-5)$ (it is free with exponents (3, 5); see [20] for a proof). Then, according to the Lemma 7.1 there is at any point $P$ a degree 4 curve with multiplicity 3 at $P$ passing through $Z$. In other words, by Theorem 5.1 $I$ fails the SLP at the range 2 and degree 2. □

Figure 2. Dual set of points of the B3 arrangement

More generally non balanced free arrangements leads to ideals that fail the SLP.

Proposition 7.4. Let $A = \{l_1, \ldots, l_{a+b+1}\}$ a line arrangement that is free with exponents $(a, b)$ such that $a \leq b, b - a \geq 2$ and $a + b$ even. The ideal $I = (l_1^{a+b}, \ldots, l_{a+b+1}^{a+b})$ fails the SLP at the range 2 and degree $\frac{a+b}{2} - 1$. 
Remark. If $a + b$ is odd we can add to $Z$ one point $P$ in general position with respect to $Z$ and we can prove in the same way that $I = (l_1^{a+b+1}, \ldots, l_{a+b+1}^{a+b+1}, (P^\vee)_{a+b+1}^{a+b+1})$ fails the SLP at the range 2 and degree $\frac{a+b}{2}$.

Proof. Let us denote by $Z = \{l_1^\vee, \ldots, l_{a+b+1}^\vee\}$ the dual set of points of $A$. Since there exists at any general point $l_1^\vee$ a curve of degree $a + 1$ passing through $Z$, the Lemma 7.1 implies that $D_0(Z)$ is unstable and Proposition 7.2 implies that $I$ fails the SLP at the range 2 and degree $\frac{a+b}{2} - 1$. □

7.1. SLP at the range 2 and Terao’s conjecture. One of the main conjecture about hyperplane arrangements (still open also for line arrangements) is the Terao’s conjecture. It concerns the free arrangements. The conjecture says that freeness of hyperplane arrangements (still open also for line arrangements) is the Terao’s Conjecture. It concerns the free arrangements. The conjecture says that freeness depends only on the combinatorics of the arrangement. Let us recall that the combinatorics of the arrangement $A = \{l_1, \ldots, l_n\}$ is determined by an incidence graph. Its vertices are the lines $l_k$ and the points $P_{i,j} = l_i \cap l_j$. Its edges are joining $l_k$ to $P_{i,j}$ when $P_{i,j} \in l_k$. We refer again to [20] for a good introduction to the subject. The Terao’s conjecture is valid not only for line arrangement but more generally for hyperplane arrangements.

Conjecture (Terao). The freeness of a hyperplane arrangement depends only on its combinatorics.

In other words an arrangement with the same combinatorics of a free arrangement is free, too.

Let us consider a free arrangement $A_0 = \{h_1, \ldots, h_n\}$ with exponents $(a, b)$ $(a \leq b)$ and let us denote by $Z_0$ its dual set of points. We assume the Terao’s conjecture is not true i.e., that there exists a non free arrangement $A = \{l_1, \ldots, l_n\}$ with the same combinatorics of $A_0$. Let us add $b - a$ points $\{M_1, \ldots, M_{b-a}\}$ in general position to $Z_0$ in order to form $\Gamma_0$ and to th dual set $Z$ of $A$ to form $\Gamma$. Then the length of both sets of points is $2b + 1$. On the general line $l$ we have

$$D_0(Z_0) \otimes \mathcal{O}_l = \mathcal{O}_l(-a) \oplus \mathcal{O}_l(-b),$$

when, since $Z$ is not free, we have a less balanced decomposition for $D_0(Z)$ (this affirmation is proved in [7]):

$$D_0(Z) \otimes \mathcal{O}_l = \mathcal{O}_l(s-a) \oplus \mathcal{O}_l(s-b), \quad s \geq 1.$$ 

It implies that

$$h^0(I_{Z_0} \otimes I_{l^\vee}^{-1}(a)) \neq 0, h^0(I_{Z_0} \otimes I_{l^\vee}^{a-1}(a)) = 0 \text{ and } h^0(I_{Z_0} \otimes I_{l^\vee}^{-b}(a+1)) \neq 0.$$ 

Then adding $b - a$ lines passing through $l^\vee$ and the $b - a$ added points we obtain

$$h^0(I_{\Gamma} \otimes I_{l^\vee}^{-1}(b)) \neq 0, h^0(I_{Z_0} \otimes I_{l^\vee}^{-1}(b)) = 0 \text{ and } h^0(I_{Z_0} \otimes I_{l^\vee}^{b-1}(b+1)) \neq 0.$$ 

The bundle $D_0(\Gamma_0)$ is balanced with splitting $\mathcal{O}_l(-b) \oplus \mathcal{O}_l(-1-b)$ and

$$D_0(\Gamma) \otimes \mathcal{O}_l = \mathcal{O}_l(1-b) \oplus \mathcal{O}_l(-1-b).$$ 

Then $D_0(\Gamma_0)$ is semistable and $D_0(\Gamma)$ is unstable. In other words the ideal

$$(l_1^b, \ldots, l_{a+b+1}^b, (M_1^\vee)^b, \ldots, (M_{b-a}^\vee)^b)$$

fails the SLP at the range 2 and degree $b - 2$ when

$$(d_1^b, \ldots, d_{a+b+1}^b, (M_1^\vee)^b, \ldots, (M_{b-a}^\vee)^b)$$

has the SLP at the range 2 and degree $b - 2$. The following conjecture written in terms of SLP is equivalent to the Terao’s conjecture on $\mathbb{P}^2$. 


Conjecture. Let $Z_0 = \{d_1^\lor, \ldots, d_{2b+1}^\lor\}$ a set of points of length $2b + 1$ in $\mathbb{P}^{2b}$ such that the ideal $I = (d_1^b, \ldots, d_{2b+1}^b)$ has the SLP at the range 2 and degree $b - 2$. Assume that $Z = \{l_1^\lor, \ldots, l_{2b+1}^\lor\}$ has the same combinatorics of $Z_0$. Then $J = (l_1^b, \ldots, l_{2b+1}^b)$ has the SLP at the range 2 and degree $b - 2$.

References


