SCALE CALCULUS AND THE SCHRÖDINGER EQUATION

by

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Abstract. — This paper is twofold. In a first part, we extend the classical differential calculus to continuous non differentiable functions by developping the notion of scale calculus. The scale calculus is based on a new approach of continuous non differentiable functions by constructing a one parameter family of differentiable functions $f(t,\epsilon)$ such that $f(t,\epsilon) \to f(t)$ when ϵ goes to zero. This lead to several new notions as representation, fractal functions and ϵ-differentiability. The basic objets of the scale calculus are left and right quantum operators and the scale operator which generalize the classical derivative. We then discuss some algebraic properties of these operators. We define a natural bialgebra, called quantum bialgebra, associated to them. Finally, we discuss a convenient geometric object associated to our study. In a second part, we define a first quantization procedure of classical mechanics following the scale relativity theory developped by Nottale. We obtain a non linear Schrödinger equation via the classical Newton's equation of dynamics using the scale operator. Under special assumptions we recover the classical Schrödinger equation and we discuss the relevance of these assumptions.

Contents

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Introduction

The origin of the fundamental incompatibility between *quantum mechanics* and Einstein's *general relativity* lies in the microscopic geometric structure of space-time. As pointed out by Greene [**27**], Feynman [**25**], Cohen-Tannoudji and Spiro ([**14**], p.131) and others, space-time is no more a *differentiable* manifold at the *atomic* scale, contrary to the assumption of general relativity.

From this fact, at least two theory have been constructed :

- the *string theory*, which implies a dimensional extension of space-time by allowing closed dimension at the Planck scale;

- the *scale relativity theory* developped by Nottale [**37**], which gives up the Einstein's assumption of the differentiability of space-time by considering what he calls a *fractal space-time*, which can be interpreted as a scale

dependant non differentiable manifold. He then extends einstein's relativity principle to scale, and developp the *scale relativity principle*.

In this article, we explore this second alternative.

Nottale [**37**] has studied what are the consequences of the abandon of the differentiability of space-time. This problem is difficult, in particular because the mathematical foundations of such a theory are not yet constructed. For example, Nottale asserts that there exists an "infinity of geodesics on a fractal space-time". This sentence is difficult to understand because we don't know what is a fractal space-time (We refer to [**20**] for a first definition of a fractal manifold and a discussion of the special scale relativity theory). Even if we identify this set to a non differentiable manifold, we don't know what is the sense of "geodesic". As a consequence, we restrict our attention to a far simpler problem, namely the consequences of the loss of differentiability of a given trajectory. A first approach is to consider that only *space* is a non differentiable manifold and to take the variable t as an absolute variable. As a consequence, trajectories of quantum particles are non differentiable curves parametrized by the time variable t. In this case, we have at least the following two consequences :

i) By Lebesgues theorem [**46**], the *lenght* of a non differentiable curve Γ is *infinite*. What does it means from the physical view-point ? That given a parameter $\epsilon > 0$, which has the sense of a resolution, the lenght L_ϵ of the curve constructed by connecting small arcs of lenght ϵ on points of Γ, goes to infinity when ϵ goes to zero. As a consequence, the role of ϵ is now fundamental as L_ϵ loes any sense when ϵ going to zero, contrary to usual differentiable curves where ϵ is only a parameter of precision and L_ϵ goes to a fixed constant L.

Nottale then introduce *fractal functions*, which are resolution (or scale) dependant functions $f(x, \epsilon)$, which converge to non differentiable functions, and a (*renormalization like*) differential equation satisfied by L_ϵ, called a *scale*

law, which gives the behaviour of $f(x, \epsilon)$ when ϵ goes to zero.

ii) The derivative along the curve has no sense. Nottale introduce a complex operator, that he calls the *scale derivative*. It takes into account the mean-backward and mean-forward derivative along the curve.

Using these tools, he gives an informal derivation of the *Schrödinger equation* from the classical *Newtonian equation of dynamics*, via a quantization procedure which follows from an extension of Einstein's relativity principle called the *scale relativity principle*.

In this paper, we developp a mathematical framework in which we can explicit the quantization procedure, which we call the *scale quantization procedure*.

The plan of the paper is as follows :

In part 1, we define a natural extension of Leibniz differential calculus which can be used on non differentiable functions in order to precise points i) and ii). We introduce the *Scale calculus*, which formalize the concept of ϵ-differentiability. In particular, we define an operator called *scale difference operator*, which is the rigourous mathematical counterpart of Nottale's scale derivative.

In part 2, we define the scale quantization procedure. We give a precise definition of the quantization map, which allows us to associate to the classical Newtonian equation of dynamics a quantized analog. This analog has the form of a generalized non linear Schrödinger equation. We then discuss how to obtain the classical Schrödinger equation.

PART I

SCALE CALCULUS

1. Introduction

Non differentiable functions, and more generally non differentiable manifolds, become more and more important in many part of mathematics and physics, like brownian motion [**24**], quantum mechanical path by Feynman and Hibbs [**26**]. Despite many works, our understanding of non differentiable functions is insatisfactory.

A great deal of efforts have been devoted to generalize, as long as possible, the classical differential calculus of Leibniz and Newton. This lead to different kind of fractional calculus (Riemann, Liouville, Weyl,...). All these fractional calculus are based on a pure analytic generalization of the Cauchy formula. As a consequence, and despite their intrinsic interest, they are difficult to interpret (in particular, from the geometrical view point).

The aim of this paper is to introduce a set of ideas, coming from physics, in order to renew our approach to non differentiable functions.

In physic, the non differentiability is not studied for itself. In the contrary, this is the effect of non differentiability with respect to a differentiable model wich is looking for. For example, as explained by Greene ([**27**], chap.5), in *superstring* theory one is lead to a new vision of spacetime because the spacetime, at the scale of atoms can not be considered differentiable at all, as in the general relativity scheme. Here, one focus on fluctuations with respect to a differentiable character.

Moreover, one usually doesn't have access via measurement, to the non differentiable object (function, manifold), but to a almost everywhere differentiable model of it. This explain also the previous remark : by measure we obtain a differentiable model, and we must see, when the precision of measure

increases, if there is no two strong fluctuations with respect to this model. For a differentiable process, the fluctuations decrease. In the case of a non differentiable process, one expect larger and larger fluctuations.

This point of view lead to several new concepts : representation of continuous non differentiable functions, fractals functions and ϵ-differentiability. The idea is to associate to each continuous non differentiable functions $f(t)$, a one parameter family of differentiable function $f(t,\epsilon)$ such that $f(t,\epsilon) \to f(t)$ when ϵ goes to zero. The next step is to give a usefull criterion which say that $f(t,\epsilon)$ is a "good" model for the function f. An important notion is then the minimal resolution wich is, more or less, the precision under which, one can't use a differentiable model without laking many features of the underlying function f. We then define quantum derivatives and the scale derivative, which reflect the non differentiable character of the underlying function f.

We discuss algebraic properties of quantum derivatives, and the scale derivative. These operators act on the set of continuous real valued functions, denoted by C^0. They are first introduced in [**4**] in order to discuss the derivation of the Schrödinger's equation from the classical Newton's equation of dynamics using Nottale's scale relativity theory [**37**].

In this paper, we construct a natural structure of bialgebra, called quantum bialgebra, using specific properties of quantum derivatives. The quantum algebra can be considered as a small deformation of a classical Hopf algebra. Although we are close to problem related to quantum groups and quasitriangular Hopf algebras introduced by Drinfeld [**22**], we stress that quantum bialgebra are new.

There is no natural bialgebra structure associated to scale derivative. This follows from the lack of a natural composition rule for scale operators that we define in this paper.

We also discuss the natural geometric object associated to a continuous non differentiable function. It turns out that this geometric object is, in the simplest case, the product of a differentiable curve Γ by a two points set $A = \{a, b\}$, so $M \times A$. This is a simple example of a noncommutative space studied by Connes [**15**], as a preliminary to his noncommutative model of the standard model.

2. About non differentiable functions

The aim of this paragraph is to develop a rigourous mathematical understanding of the idea of fluctuation with respect to a differentiable model for continuous non differentiable functions.

2.1. Representation theory and fractals functions. — Here, we introduce two dual notions : representation of non differentiable functions and fractals functions. Representation are well suited to develop a mathematical understanding of non differentiable function based on differentiable functions. Fractals functions takes their origin in physical problems.

In the following, we denote by C^0 the set of continuous real valued functions, and by C^1 the set of differentiable real valued functions.

Definition 2.1. — *Let $f \in C^0$, a representation of f is a one parameter family of operator S_ϵ, defined by*

$$S_\epsilon : \begin{array}{ccc} C^0 & \longrightarrow & C^1 \\ f & \longmapsto & S_\epsilon(f) = f_\epsilon, \end{array} \tag{1}$$

and such that the differentiable functions $f_\epsilon \in C^1$ converge, in C^0-topology, toward f when ϵ goes to zero.

A basic example is obtain by approximating f by mean functions f_ϵ defined by $f_\epsilon(t) = (1/2\epsilon) \displaystyle\int_{t-\epsilon}^{t+\epsilon} f(s)ds$.

More generally, we can consider a *smoothing function* $\Phi(s;t,\epsilon)$ depending on two parameters, t and ϵ, satisfying the normalization

$$\int_{-\infty}^{\infty} \Phi(s;t,\epsilon)ds = 1. \tag{2}$$

For any continuous function, we define a representation by

$$f_\epsilon(t) = \int_{-\infty}^{\infty} \Phi(s;t,\epsilon)f(s)ds. \tag{3}$$

In practice, we have never access to f. An idea is to define f via a family of functions which are not functionally dependent of f like in (1). We are lead to the notion of *fractal functions*, first introduced by Nottale [**37**] (see also [**18**]).

Definition 2.2. — *A fractal function is a parametrized function of the form $f(t,\epsilon)$, depending on $\epsilon > 0$, such that :*

i) For all $\epsilon > 0$, the function $f(t,\epsilon) \in C^1$, except at a finite number of points,

ii) There exists an everywhere non differentiable function $f(t)$ such that $f(t,\epsilon)$ converge to $f(t)$ when ϵ goes to zero.

The main difference between definition 2.1 and 2.2 is that for fractal functions, one usually doesn't know an explicit form of the limit function f, We only require an existence result. Moreover, the set of functions $f(t,\epsilon)$ doesn't refer to the limit f in its definition, which is closest to the *measurement* process in physical experiment.

2.1.1. Examples of fractal functions : Nottale's functions and iteration of affine systems. — The basic example of fractal functions is *Nottale's functions* introduced in [**37**] :

For all $\epsilon > 0$, and for all $0 < \mu < \epsilon$,

$$x(t,\epsilon) = \int \Phi_{\epsilon,\mu}(t,y)x(y,\mu)dy, \tag{4}$$

where $\Phi_{\epsilon,\mu}(x,y)$ is a differentiable function such that

$$\int_{-\infty}^{\infty} \Phi_{\epsilon,\mu}(x,y)dy = 1, \ \forall\, x \in \mathbf{R}, \tag{5}$$

called a smoothing function.

Definition 2.3. — *Let $\Phi_{\epsilon,\mu}$ be a smoothing function satisfying (5). We denote by $\mathcal{N}(\Phi_{\epsilon,\mu})$, and we call Nottale's set associated to $\Phi_{\epsilon,\mu}$, the set of functions defined by (4).*

We refer to [**18**] for basic properties of this set of functions, in particular for a usefull equivalence relation.

An interesting example of fractal functions for which the limit function is not explicit is given by iteration of *affine systems* [**46**].

An affine map in $\mathbb{R}^2$, with a coordinates system (x,y), is a map of the form

$$F\begin{pmatrix} x \\ y \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix} + T, \tag{6}$$

where M is a 2×2 matrix and T is a translation vector.

An *affine system* is given by :

i) a positive integer $N \geq 2$,

ii) $N+1$ points of $\mathbb{R}^2$, $A = A_1, \ldots, A_{N+1} = B$. We denote $A_i = (x_i, y_i)$, $x_1 = a$, $x_{N+1} = b$. We assume that

$$a = x_1 < x_2 < \cdots < x_{N+1} = b.$$

iii) N affine map $F_1, \ldots, F_N$ such that

$$F_i(AB) = A_i A_{i+1}.$$

We denote by F the map defined by

$$F(E) = \bigcup_{i=1}^{N} F_i(E). \tag{7}$$

Let z_0 be the affine function on $[a,b]$ whose graph Γ_0 is the segment AB. The image $\Gamma_1 = F(\Gamma)$ is the graph of a continuous function, which is affine. For all n, we define the continuous function z_n whose graph is $\Gamma_n = F(\Gamma_{n-1})$. Following ([**46**], p.175), the sequence of functions $(z_n)_{n\in\mathbb{N}}$ converge uniformly to a continuous function z_∞ such that $F(\Gamma_\infty) = \Gamma_\infty$.

2.2. ϵ-differentiability and minimal resolution. —

2.2.1. Minimal resolution : formal idea. — In our point of view, non differentiability is always studied via a one parameter familly of differentiable functions f_ϵ. A basic question is the following :

When ϵ goes to zero, can we find a value ϵ_0 such that for $\epsilon > \epsilon_0$, we can assume that the limit function is differentiable and for $\epsilon > \epsilon_0$, we are sure that the limit function is non differentiable ?

It is equivalent to ask if the non differentiable character of a function can be detected via its approximation.

If we can find a quantity $\epsilon(f)$ of this kind, then we call it *minimal resolution*. It is the best order of approximation under which non differentiable effects must be taken into account.

In the following we give two ways in order to defined a minimal resolution, by taking two different effects of non differentiability.

2.2.2. First approach. — A basic properties of differentiable functions is that the quantities

$$\bigtriangledown_+^\epsilon f(t) = \frac{f(t+\epsilon)-f(t)}{\epsilon}, \text{ and } \bigtriangledown_-^\epsilon f(t) = \frac{f(t)-f(t-\epsilon)}{\epsilon}, \tag{8}$$

keep sense when ϵ goes to zero and are equal.

As a consequence, the following quantity

$$a_\epsilon f(t) = \left| \frac{f(t+\epsilon) + f(t-\epsilon) - 2f(t)}{\epsilon} \right|, \tag{9}$$

converges to zero when ϵ goes to zero.

The underlying idea is that the two representations of a function f, given by the forward and backward mean function, defined as

$$f_\epsilon^+(t) = (1/2\epsilon) \int_t^{t+\epsilon} f(s)ds \quad \text{and} \quad f_\epsilon^-(t) = (1/2\epsilon) \int_{t-\epsilon}^{t} f(s)ds, \tag{10}$$

respectively, must have derivatives which coincide when ϵ goes to zero.

This remark allows us to introduce the following notion of ϵ-h-differentiability :

Definition 2.4. — *Let $h > 0$ be a given real number. A function $f \in C^0$ is says to be ϵ-h-differentiable at point t, if*

$$a_\epsilon f(t) < h. \tag{11}$$

We can detect the non differentiable character of a function by investigating its ϵ-h differentiability. Precisely, we define the notion of minimal resolution :

Definition 2.5. — *Let $h > 0$ be a given real number and $f \in C^0$. The h-minimal resolution of f at point t, denoted $\epsilon(f,h)(t)$ is defined as $\inf_\epsilon\{a_\epsilon f(t) < h\}$.*

Of course, if for a given h, the h-minimal resolution is non zero, then f is non differentiable.

Remark 2.1. — *The reverse is wrong, as proved by the following example :*

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f : \begin{cases} t\cos(1/t) & \text{if} \quad t \neq 0, \\ 0 & \text{if} \quad t = 0. \end{cases} \tag{12}$$

This function is derivable for all $t \neq 0$. Then, we have $\epsilon(f,h)(t) = 0$ for all $t \neq 0$. This function is not derivable at $t = 0$. However, we have $a_\epsilon(f,h)(0) =$

0 for all $\epsilon > 0$ by parity. As a consequence, $\epsilon(f,h) = 0$, even if f is not derivable on $\mathbb{R}$.

For all $\alpha \in]0,1[$, we denote by C^α the set of continuous real valued functions, defined on $[0,1]$ such that the quantity

$$| f |_\alpha = \sup_{0 \leq x \neq y \leq 1} \frac{| f(x) - f(y) |}{| x - y |^\alpha}, \tag{13}$$

is finite (Hölderian functions of order α). Then, we have :

Lemma 2.1. — *Let $0 < \alpha < 1$ and $f \in C^\alpha$. For all $t \in]0,1[$, and all $h > 0$, the h-minimal resolution of f at point t satisfies*

$$\epsilon(f,h)(t) \leq \left(\frac{h}{2 \mid f \mid_\alpha} \right)^{1/\alpha - 1}. \tag{14}$$

Remark 2.2. — *In this example, the minimal resolution depends on $| f |^\alpha$. As a consequence, for a quantum mechanical path, we expect that the minimal resolution depends on the momentum of the particle. This is indeed the case in Nottale's theory* [**37**]*, where the minimal resolution is related to the De Broglie lenght of the particle.*

A global order of h-minimal resolution can be defined.

Definition 2.6. — *Let $h > 0$ be given and $f \in C^0$. The h-minimal resolution of f, denoted $\epsilon(f,h)$, is defined by $\epsilon(f,h) = \sup_{t \in \mathcal{D}f} \epsilon(f,h)(t)$, where $\mathcal{D}f$ is the definition domain of f.*

In this definition it is important to take the sup of h-minimal resolution of f at point t. As a consequence, if f is differentiable on a small set of point, the h-minimal resolution is however non zero.

Remark 2.3. — *We have $\epsilon(f + c, h) = \epsilon(f,h)$ for all $c \in \mathbb{R}$. But, we have $\epsilon(\lambda f, h) \neq \epsilon(f,h)$ for all $\lambda \neq 1$, contrary to the case of* [**4**]*. This inequality is related to the fact that a changing momentum induce a change of regularity for the curve. A physical consequence, is that the minimal resolution must depends on the momentum.*

The connexion to representation theory is done through the backward and forward mean functions, f_ϵ^σ, $\sigma = \pm$, introduced before. Indeed, backward and forward mean functions are differentiable functions. They can be used as a classical representation of a given continuous function f. However, if f admits a non zero h minimal resolution, this means that these functions are not sufficient to capture the complete local behaviour of f as long as $\epsilon < \epsilon(f, h)$. Of course, this notion depends on h. In physical problems, the constant h must corresponds to a universal constraint, like Heisenberg constraint in quantum mechanics.

2.2.3. Order of divergence. — The previous quantity $a_\epsilon f(t)$ gives a criterion distinguishing differentiable and non differentiable functions based on the fact that in some case, the left and right derivative of a non differentiable function (when they exist) are different. Of course, the general case is far more complicated. A complete characterization of differentiability is given by the following :

Let f be a continuous real function. We denote

(15)
$$D^\sigma f(t) = \lim_{\epsilon \to 0^\sigma} \sup \frac{f(t+\epsilon) - f(t)}{\epsilon}, \text{and } D_\sigma f(t) = \lim_{\epsilon \to 0^\sigma} \inf \frac{f(t+\epsilon) - f(t)}{\epsilon},\ \sigma = \pm.$$

These quantities are always well defined and belong to $\bar{\mathbb{R}}$. By ([**45**],[**31**],p.319), we know that f is differentiable at point t if and only if

$$D^+ f(t) = D^- f(t) = D_+ f(t) = D_- f(t) \in \mathbb{R}. \tag{16}$$

By [**45**], we know that the set of continuous functions for which the left and right derivative exist and are distinct is of zero measure. By considering the set of non differentiable functions, we then deduce that the left and right derivative doesn't exist on a set of full measure. More precisely, we have two case : either $D^\sigma f(t)$ and $D_\sigma f(t)$ exists but are different, or these quantities diverge with ϵ going to zero, for $\sigma = \pm$.

We denote by $\Delta_\mu^\sigma f(t)$, $\sigma = \pm$, the quantity

$$\Delta_\mu^\sigma f(t) = \sigma \frac{f(t+\sigma\mu) - f(t)}{\mu},\ \sigma = \pm, \tag{17}$$

and for $\epsilon > 0$, we denote

$$\bar{d}_\epsilon^\sigma f(t) = \sup_{\mu \leq \epsilon} \Delta_\mu^\sigma f(t),\ \underline{d}_\epsilon^\sigma f(t) = \inf_{\mu \leq \epsilon} \Delta_\mu^\sigma f(t). \tag{18}$$

A better object to measure the loss of differentiability is the following quantity

$$A_\epsilon f(t) = \begin{array}{l} \inf_{\epsilon>0} \left\{ \mid \bar{d}_\epsilon^+ f(t) - \underline{d}_\epsilon^+ f(t) \mid < h, \mid \bar{d}_\epsilon^- f(t) - \underline{d}_\epsilon^- f(t) \mid < h, \right. \\ \left. (1 - 1_{D^+=D_+})(1 - 1_{D^-=D_-}) \mid \bar{d}_\epsilon^+ f(t) - \bar{d}_\epsilon^- f(t) \mid < h \right\}. \end{array} \tag{19}$$

when ϵ goes to zero.

In the Hölderian case, i.e. $f \in C^\alpha$, $0 < \alpha < 1$, this quantity diverge as $\epsilon^{\alpha-1}$. As a consequence, a possible extension of the notion of minimal resolution can be obtain by the following.

Definition 2.7. — *Let $h > 0$, we call minimal resolution the following quantity*

$$\epsilon(f,h) = \inf_{\epsilon>0} \{ A_\epsilon f(t) < h \}. \tag{20}$$

When the left and right derivative exist, or diverge, the previous quantity reduce to the minimal resolution defined in 2.6. However, this quantity can distinguish the case where left and right derivatives are not defined but not infinite.

2.3. Scale law. — For an everywhere non-differentiable function f, the lenght L_ϵ of the graph of the mean function f_ϵ goes to infinity when ϵ goes to 0. Of course, this property can't be used as a definition of an everywhere non differentiable function. We can find curve which are rectifiable with infinite lenght (see [**6**]). We want to quantify this divergence. A first idea is to find a differential equation which gives the behaviour of L_ϵ with respect to ϵ like in ([**18**],[**6**]). However, this is difficult because L_ϵ is not differentiable with respect to ϵ in general. In this section, we define a less rigid definition of scale law and we discuss its properties.

2.3.1. Definition. — Let f be an everywhere non differentiable continuous function on the interval $I = [0, 1]$. For $\epsilon > 0$, we denote by f_ϵ the mean function associated to f on I, and $\mathcal{L}_\epsilon$ the lenght of its graph.

Definition 2.8. — *We say that f satisfies a scale law if there exist functions $l_\epsilon > 0$ and $L_\epsilon > 0$ such that*

$$l_\epsilon \leq \mathcal{L}_\epsilon \leq L_\epsilon, \tag{21}$$

satisfying

$$\mathcal{L}_\epsilon = O(l_\epsilon),\ \mathcal{L}_\epsilon = O(L_\epsilon), \tag{22}$$

and for which there exist a function $E : \mathbb{R} \to \mathbb{R}$, such that

$$\frac{dl_\epsilon}{d\ln\epsilon} = E(l_\epsilon, \ln\epsilon),\ \frac{dL_\epsilon}{d\ln\epsilon} = E(L_\epsilon, \ln\epsilon). \tag{23}$$

The function E is called a scale law.

Of course, when one knows a scale law of f, we deduce a speed of drift for $\mathcal{L}_\epsilon$.

Basic examples of scale laws are given by

i) For $a > 0$, $E(x, t) = a$,

ii) For $b > 0$, $E(x, t) = bx$.

For i), we obtain graph with logarithmic drift of order $\mid a \mid \ln(1/\epsilon)$. For ii), we obtain classical power law drift of order $1/\epsilon^b$.

Using this notion, we are lead to two different kind of problems :

i) Let E be a given function. Find the set of function $\mathcal{E}(E)$ such that for all $f \in \mathcal{E}(E)$, a scale law of f is E.

ii) Let $\mathcal{E}$ be a given set of functions. Find, if it exits, a scale law for each $f \in \mathcal{E}$.

The first problem is equivalent to estimate a given class of functions by the speed of drift of $\mathcal{L}_\epsilon$. This point is difficult and discussed in [**6**].

The second one is more natural and discussed in the following.

2.3.2. Scale law of Hölderian functions. — We first define an important class of continuous functions.

Definition 2.9. — *We denote by $H^\alpha(c, C)$ the set of real valued continuous functions f such that for all $\epsilon > 0$ sufficiently small, and $\mid t - t' \mid < \epsilon$, we have*

$$c\epsilon^\alpha \leq \mid f(t) - f(t') \mid \leq C\epsilon^\alpha. \tag{24}$$

The set H^α corresponds to continuous functions which are Hölder and inverse Hölder of exponent α.

Example. Let $0 < \alpha < 1$ and $g(t)$ be the function of period 1 defined on $[0, 1]$ by

$$g(t) = \begin{cases} 2t, & if\ 0 \leq t \leq 1/2, \\ 2 - 2t, & if\ 1/2 \leq t \leq 1. \end{cases} \tag{25}$$

The Knopp or Takagi function, defined by

$$K(t) = \sum_{n=0}^{\infty} 2^{-n\alpha} g(2^n t), \tag{26}$$

belongs to H^α (see [**46**],§.13.1).

We use notations from definition 2.8

Theorem 2.1. — *Let $0 < \alpha < 1$ and $f \in H^\alpha(c, C)$ defined on an open interval $U \subset \mathbb{R}$ such that $I = [0, 1] \subset U$. For all $\epsilon > 0$, we define*

$$l_\epsilon = \epsilon^{\alpha-1}\sqrt{\epsilon^{2(1-\alpha)} + c^2},\ L_\epsilon = \epsilon^{\alpha-1}\sqrt{\epsilon^{2(1-\alpha)} + C^2}. \tag{27}$$

A scale law for f is then given by

$$E(y,t) = (\alpha - 1)(y - 1/y). \tag{28}$$

Proof. We have

$$\mathcal{L}_\epsilon = \frac{1}{2\epsilon}\int_0^1 \sqrt{4\epsilon^2 + (f(x+\epsilon) - f(x-\epsilon))^2}dx. \tag{29}$$

As $f \in H^\alpha$, we have

$$4c^2\epsilon^{2\alpha} \leq (f(x+\epsilon) - f(x-\epsilon))^2 \leq 4C^2\epsilon^{2\alpha}. \tag{30}$$

As a consequence, we obtain

$$\epsilon^{\alpha-1}\sqrt{\epsilon^{2(1-\alpha)} + c^2} \leq \mathcal{L}_\epsilon \leq \epsilon^{\alpha-1}\sqrt{\epsilon^{2(1-\alpha)} + C^2}. \tag{31}$$

We deduce

$$l_\epsilon \leq \mathcal{L}_\epsilon \leq L_\epsilon, \tag{32}$$

for ϵ sufficiently small.

By differentiating L_ϵ with respect to ϵ, we obtain

$$\frac{dL_\epsilon}{d\epsilon} = \frac{\alpha - 1}{\epsilon}\left[L_\epsilon - \frac{1}{L_\epsilon}\right]. \tag{33}$$

Using

$$\frac{dL_\epsilon}{d\ln\epsilon} = \epsilon\frac{dL_\epsilon}{d\epsilon}, \tag{34}$$

we obtain the scale function $E(y,t) = (\alpha - 1)(y - 1/y)$. We verify that l_ϵ satisfies the same differential equations. □

The previous result is best analyzed in term of the new variables

$$x_\epsilon = 1/l_\epsilon,\ \mathcal{X}_\epsilon = 1/\mathcal{L}_\epsilon,\ X_\epsilon = 1/L_\epsilon. \tag{35}$$

When ϵ goes to 0, we have x_ϵ, X_ϵ and $\mathcal{X}_\epsilon$ which go to 0. Moreover, the scale law for these new functions is

$$\frac{dx}{dt} = (1 - \alpha)(x - x^3). \tag{36}$$

Indeed, by making the change of variables $x = 1/y$ in the scale law

$$\frac{dy}{dt} = (\alpha - 1)(y - 1/y), \tag{37}$$

and using the relation

$$\frac{dx}{dt} = -(1/y^2)\frac{dy}{dt}, \tag{38}$$

we obtain the result.

The classical linearization theorem of Poincaré ([**2**],[**30**]) allows us to find, in a neighborhood of $x = 0$, an analytic change of variables $z = h(x)$, such that the differential equation (36) is transformed into

$$\frac{dz}{dt} = (1-\alpha)z. \tag{39}$$

The set of Hölderian functions H^α induces, up to analytic changes of variables, the linear scale law (39).

Remark 2.4. — *In Galilean scale relativity, the set of functions which admit a linear scale law allows us to define the Djinn variable (see* [**18**],*§.3.2.5 and §.3.3).*

2.3.3. Non uniform Hölderian functions. — In this section, we consider non uniform Höderian functions.

Definition 2.10. — *Let* $\alpha(t) : \mathbb{R} \to]0,1[$. *We denote by* $H^{\alpha(.)}$ *the set of continuous Hölderian functions satisfying, for all* $h > 0$ *sufficiently small,*

$$ch^{\alpha(t)} <\mid f(t+h) - f(t) \mid< Ch^{\alpha(t)}, \tag{40}$$

where $c > 0$, $C > 0$ *are constants.*

For $H^{\alpha(.)}$ functions, we have not been able to derive a scale law. We then introduce a weak notion :

Definition 2.11. — *We say that* f *admits weak-scale laws, if there exists* l_ϵ *and* L_ϵ *such that*

$$l_\epsilon \leq \mathcal{L}_\epsilon \leq L_\epsilon, \tag{41}$$

satisfying

$$\mathcal{L}_\epsilon = O(l_\epsilon), \quad \mathcal{L}_\epsilon = O(L_\epsilon), \tag{42}$$

and for which we can find two functions $E_- : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *and* $E_+ : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *such that*

$$\frac{dl_\epsilon}{d\ln\epsilon} = E_-(l_\epsilon, \ln\epsilon), \quad \frac{dL_\epsilon}{d\ln\epsilon} = E_+(L_\epsilon, \ln\epsilon). \tag{43}$$

For all $\epsilon > 0$, we define

$$\alpha_i(\epsilon) = \alpha(i/\epsilon i + 1/\epsilon), \tag{44}$$

with $i = 0, \ldots, [1/\epsilon]$, where $[x]$ denotes the integer part of x.

We define

$$\gamma(\epsilon) = \min_{i=0,\ldots,[1/\epsilon]} \alpha_i(\epsilon),\ \beta(\epsilon) = \max_{i=0,\ldots,[1/\epsilon]} \alpha_i(\epsilon). \tag{45}$$

Theorem 2.2. — *Let* $f \in H^{\alpha(.)}$. *For all* $\epsilon > 0$ *sufficiently small we assume that the exponents (45) are differentiable functions with respect to* ϵ. *Then,* f *admits for weak-scale laws*

$$E_-(x,t) = (1-\alpha-t\gamma')(x-x^3), \quad E_+(x,t) = (1-\alpha-t\beta')(x-x^3). \tag{46}$$

3. Scale calculus

In [**4**], we introduce the notion of *quantum derivatives*. In this section, we give a less rigid definition, which allow us to discuss more easily algebraic properties of these operators.

3.1. Left and right quantum difference operators. — Let $h > 0$ be given. If f possesses a non zero h-minimal resolution, then for $\epsilon < \epsilon(f,h)$, one must take into account the non differentiable character of f with respect to its forward and backward mean representations. A possible way to do it, is to say that the backward and forward derivatives of f at ϵ carry different informations on the local behaviour of f. The idea of quantum derivatives formalize this idea.

Definition 3.1. — *Let* $h > 0$, *and* f *be a continuous, real valued fonction.*

If $\epsilon(f,h) > 0$, for all $\epsilon(f,h) > \epsilon > 0$, we define left and right quantum difference operators of f at point t, the quantities

$$\bigtriangledown_+^\epsilon f(t) = \frac{f(t+\epsilon) - f(t)}{\epsilon},\ \bigtriangledown_-^\epsilon f(t) = \frac{f(t) - f(t-\epsilon)}{\epsilon}, \tag{47}$$

respectively.

If $\epsilon(f,h) = 0$, and f is differentiable then

$$\Delta_+^0 f(t) = \Delta_-^0 f(t) = f'(t). \tag{48}$$

Remark 3.1. — *1. We can give a more rigid definition of quantum derivatives by fixing the ϵ to be the minimal resolution of the function considered for a given $h > 0$.*

2. By remark 2.1, we can't extend the definition of the left and right quantum operators when $\epsilon(f,h) = 0$ in order to cover the set of non differentiable function with zero minimal resolution.

3.2. The scale difference operator. — The scale difference operator, first introduced in [**4**] following Nottale's work [**37**], is intended to summarize the information given by quantum difference operator, need to perform the local analysis of a given non differentiable function.

Definition 3.2. — *Let $h > 0$, and f be a continuous function such that $\epsilon(f,h) \geq \epsilon > 0$. The ϵ-scale difference operator of f at point t, is a complex operator, denoted by $\Box_\epsilon/\Box t$, defined by*

$$\frac{\Box_\epsilon f}{\Box t}(t) = \frac{1}{2}(\bigtriangledown_+^\epsilon f(t) + \bigtriangledown_-^\epsilon f(t)) - i\frac{1}{2}(\bigtriangledown_+^\epsilon f(t) - \bigtriangledown_-^\epsilon f(t)),\ i^2 = -1. \tag{49}$$

When $\epsilon = 0$, and f is differentiable, we have the following usefull property :

Lemma 3.1 **(Gluing)**. — *Let f be a differentiable function. Then, we have*

$$\frac{\Box f}{\Box t} = \frac{df}{dt}. \tag{50}$$

Moreover, if we denote by $f_\epsilon(t)$ the mean function $f_\epsilon(t) = (1/2\epsilon)\int_{t-\epsilon}^{t+\epsilon} f(s)ds$, we have that

$$\mathrm{Re}(\frac{\Box_\epsilon f}{\Box t}(t)) = (f_\epsilon)'(t), \tag{51}$$

where *Re* denote the real part of a complex number. The imaginary part of $\Box_\epsilon f/\Box t$ is the fluctuation of the forward mean function with respect to the backward mean function.

3.3. Main result. — We denote by H^α the set of real valued functions which are Hölder and inverse Hölder of exponent α, which mean that for all ϵ sufficiently small, and $\mid t' - t \mid \leq \epsilon$, there exists $c > 0$ and $C > 0$, such that

$$c\epsilon^\alpha \leq \mid f(t') - f(t) \mid \leq C\epsilon^\alpha. \tag{52}$$

An important theorem, from the point of view of the scale relativity, is the following :

Theorem 3.1. — *Let $h > 0$, $f(x,t)$ be a C^n function, $x(t)$ a continuous function such that $\epsilon(x,h) > 0$. Then, for $\epsilon(x,h) \geq \epsilon > 0$, ϵ sufficiently small, we have*

$$\frac{\Box_\epsilon f(x(t),t)}{\Box t} = \frac{\partial f}{\partial t} + \sum_{j=1}^{n} \frac{1}{j!}\frac{\partial^j f}{\partial x^j}(x(t),t)\epsilon^{j-1}a_{\epsilon,j}(t) + o(\epsilon^{1/n}), \tag{53}$$

where

$$a_{\epsilon,j}(t) = \frac{1}{2}\left[\left((\Delta_+^\epsilon x)^j - (-1)^j(\Delta_-^\epsilon x)^j\right) - i\left((\Delta_+^\epsilon x)^j + (-1)^j(\Delta_-^\epsilon x)^j\right)\right] .. \tag{54}$$

The proof follows from the following lemma :

Lemma 3.2. — *Let $f(x,t)$ be a real valued function of class C^{n+1}, $n \geq 3$, and let $X(t)$ be a continuous real valued function of class $H^{1/n}$. For ϵ sufficiently small, the right and left derivatives of $f(X(t),t)$ are given by*

$$\bigtriangledown_\sigma^\epsilon f(X(t),t) = \frac{\partial f}{\partial t}(X(t),t) + \sigma\sum_{i=1}^{n}\frac{1}{i!}\frac{\partial^i f}{\partial x^i}(X(t),t)\epsilon^{-1}(\sigma\epsilon\bigtriangledown_\sigma^\epsilon X(t))^i + o(\epsilon^{1/n}), \tag{55}$$

for $\sigma = \pm$.

For $n = 2$, we obtain the so-called Itô formula :

$$\begin{aligned} \bigtriangledown_\sigma^\epsilon f(X(t),t) &= \frac{\partial f}{\partial t}(X(t),t) + \frac{\partial f}{\partial x}(X(t),t) \bigtriangledown_\sigma^\epsilon X(t) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X(t),t)\epsilon(\bigtriangledown_\sigma^\epsilon X(t))^2 + o(\epsilon^{1/2}), \end{aligned} \tag{56}$$

Proof. This follows from easy computations. First, we remark that, as $X(t) \in H^{1/n}$, we have $\mid \epsilon \bigtriangledown_\sigma^\epsilon X(t) \mid = o(\epsilon^{1/n})$. Moreover,

$$f(X(t+\epsilon), t+\epsilon) = f(X(t) + \epsilon \bigtriangledown_+^\epsilon X(t), t+\epsilon). \tag{57}$$

By the previous remark, and the fact that f is of order C^{n+1}, we can make a Taylor expansion up to order n with a controled remainder.

$$\begin{aligned} f(X(t+\epsilon),t+\epsilon) &= f(X(t),t) + \sum_{k=1}^n \frac{1}{k!} \sum_{i+j=k} (\epsilon \bigtriangledown_+^\epsilon X(t))^i \epsilon^j \frac{\partial^k f}{\partial^i x \partial^j t}(X(t),t) \\ &\quad + o((\epsilon \bigtriangledown_+^\epsilon X(t))^{n+1}). \end{aligned} \tag{58}$$

As a consequence, we have

$$\epsilon \bigtriangledown_+^\epsilon f(X(t),t) = \sum_{k=1}^n \frac{1}{k!} \sum_{i+j=k} (\epsilon \bigtriangledown_+^\epsilon X(t))^i \epsilon^j \frac{\partial^k f}{\partial^i x \partial^j t}(X(t),t) + o((\epsilon \bigtriangledown_+^\epsilon X(t))^{n+1}). \tag{59}$$

By selecting terms of order less or equal to one in ϵ in the right of this equation, we obtain

$$\begin{aligned} \epsilon \bigtriangledown_+^\epsilon f(X(t),t) &= \epsilon \left[\frac{\partial f}{\partial t}(X(t),t) + \sum_{i=1}^n \frac{1}{i!} \frac{\partial^i f}{\partial x^i}(X(t),t) \epsilon^{-1} (\epsilon \bigtriangledown_+^\epsilon X(t))^i \right] \\ &\quad + o(\epsilon^2 \bigtriangledown_+^\epsilon X(t)). \end{aligned} \tag{60}$$

Dividing by ϵ, we obtain the lemma. □

3.4. The complex case. — In part II, we need to apply the scale operator to *complex valued* functions. We extend the definition of $\Box_\epsilon / \Box t$ in order to cover this case.

In the following, if $z \in \mathbb{C}$, we denote by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, the real and imaginary part of z.

Definition 3.3. — *Let $h > 0$ and $\mathcal{C}(t)$ be a complex valued function. We denote $\mathcal{C}(t) = \mathcal{C}_r(t) + i\mathcal{C}_m(t)$, where $\mathcal{C}_r(t) = \mathrm{Re}\mathcal{C}(t)$ and $\mathcal{C}_m(t) = \mathrm{Im}\mathcal{C}(t)$. We define $\Box_\epsilon\mathcal{C}/\Box t$ by*

$$\frac{\Box_\epsilon\mathcal{C}}{\Box t} = \frac{\Box_\epsilon\mathcal{C}_r}{\Box t} + i\frac{\Box_\epsilon\mathcal{C}_m}{\Box t}, \tag{61}$$

for $0 < \epsilon < \min(\epsilon(\mathcal{C}_r, h), \epsilon(\mathcal{C}_m, h))$.

Remark 3.2. — *The extension of the scale calculus to complex valued functions is not trivial (as it mixes complex terms in a complex operator).*

We then have :

Lemma 3.3. — *Let $h > 0$ and $C(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a C^n complex valued function. Let $x(t)$ be a continuous function such that $\epsilon(x,h) > 0$. We denote $\mathcal{C}(t) = C(x(t),t)$. Then, for $0 < \epsilon \leq \epsilon(x,h)$ sufficiently small, we have*

$$\frac{\Box_\epsilon\mathcal{C}}{\Box t} = \frac{\partial C}{\partial t} + \frac{\Box_\epsilon x}{\Box t}\frac{\partial C}{\partial x} + \sum_{j=2}^{n}\frac{1}{j!}a_{\epsilon,j}(t)\frac{\partial^j C}{\partial x^j}\epsilon^{j-1} + o(\epsilon^{1/n}), \tag{62}$$

where

$$a_{\epsilon,j}(t) = \frac{1}{2}[(\Delta_+^\epsilon x)^j - (-1)^j(\Delta_-^\epsilon x)^j] - i\frac{1}{2}[(\Delta_+^\epsilon x)^j + (-1)^j(\Delta_-^\epsilon x)^j],\ j = 2,\dots,n. \tag{63}$$

4. Algebraic properties of quantum difference operators

In the following, we denote by C^0 the set of continuous real valued functions.

Lemma 4.1. — *For all $\epsilon > 0$, and all $f \in C^0$ and $g \in C^0$, we have :*

i) $\bigtriangledown_\sigma^\epsilon(f+g) = \bigtriangledown_\sigma f + \bigtriangledown_\sigma g$, $\sigma = \pm$,

ii) For all $\lambda \in \mathbb{R}$, $\bigtriangledown(\lambda f) = \lambda \bigtriangledown_\sigma f$, $\sigma = \pm$.

The proof is straightforward and let to the reader.

Our main is to compare quantum derivatives to classical derivatives. The main properties of classical derivatives is the so called *Leibniz rule*, which says that, $(fg)' = f'g + fg'$. In our case, we have a more complicated formula.

Lemma 4.2. — *For all $\epsilon > 0$, and all $f \in C^0$, $g \in C^0$, we have for $\sigma = \pm$,*

$$\nabla^\epsilon_\sigma(fg)(x) = \nabla^\epsilon_\sigma f(x)g(x) + f(x)\,\nabla^\epsilon_\sigma\, g(x) + \sigma\epsilon\,\nabla^\epsilon_\sigma\, f(x)\,\nabla^\epsilon_\sigma\, g(x). \tag{64}$$

Proof. Easy computations lead to the following formulas :

$$\begin{array}{rcl} \nabla^\epsilon_+(fg)(x) & = & \nabla^\epsilon_+ f(x)g(x+\epsilon) + f(x)\,\nabla^\epsilon_+\, g(x), \\ \nabla^\epsilon_-(fg)(x) & = & \nabla^\epsilon_- f(x)g(x) + f(x-\epsilon)\,\nabla^\epsilon_-\, g(x). \end{array} \tag{65}$$

By definition of the quantum derivatives, we have

$$f(x+\epsilon) = f(x) + \epsilon\,\nabla^\epsilon_+\, f(x), \quad g(x-\epsilon) = g(x) - \epsilon\,\nabla^\epsilon_-\, g(x). \tag{66}$$

By replacing $f(x+\epsilon)$ and $g(x-\epsilon)$ in (65), we obtain (64). □

Of course, one can derived others formulas. However, equation (64) is the most symetric one. Here, we give another expression.

For all ϵ, we denote by $\tau_\epsilon : C^0 \to C^0$ the classical translation, defined by

$$\tau_\epsilon(f)(x) = f(x+\epsilon),\ \forall x. \tag{67}$$

We have the following lemma :

Lemma 4.3. — *For all ϵ, we have $\nabla^\epsilon_- \circ \tau_\epsilon = \nabla^\epsilon_+$ and $\tau_\epsilon \circ \nabla^\epsilon_\sigma = \nabla^\epsilon_\sigma \circ \tau_\epsilon$.*

As a consequence, we obtain the following version of lemma 4.2 :

Lemma 4.4. — *For all $\epsilon > 0$, and all $f \in C^0$, $g \in C^0$, we have*

$$\begin{array}{rcl} \nabla^\epsilon_+(fg)(x) & = & \nabla^\epsilon_+ f(x)h_\epsilon(x) + f(x)\,\nabla^\epsilon_-\, h_\epsilon(x), \\ \nabla^\epsilon_-(fg)(x) & = & \nabla^\epsilon_- f(x)v_\epsilon(x) + f(x)\,\nabla^\epsilon_+\, v_\epsilon(x), \end{array} \tag{68}$$

where $h_\epsilon = \tau_\epsilon \circ g$ and $v_\epsilon = \tau_{-\epsilon} \circ g$.

In the following, we discuss what are the fundamental differences between quantum derivatives and classical derivations.

4.1. Derivations. — We recall that an operator D on an abstract algebra $(A,.)$, is a derivation if for all $(f,g) \in A^2$, it satisfies the Leibniz relation $D(fg) = Df.g + f.Dg$. We refer to Jacobson ([**32**], chap.1, §.2, p.7-8) for more details.

We denote by $Der(A)$ the set of derivations on A. $Der(A)$ is a vector space, but not an algebra. However, by posing $[D_1, D_2] = D_1D_2 - D_2D_1$, the usual Lie Bracket, the set $(Der(A), [.,.])$ is a Lie algebra.

We denote by $\mathbb{R}\langle\langle Der(A)\rangle\rangle$ the ring of formal power series on the alphabet $Der(A)$. We can defined a coalgebra structure on $\mathbb{R}\langle\langle Der(A)\rangle\rangle$. We refer to Bourbaki ([**9**],chap.3) for more details about coalgebras and bialgebras.

Let $u : \mathbb{R}\langle\langle Der(A)\rangle\rangle \to \mathbb{R}$ be the homomorphism associating to each serie its constant term.

For each $D \in Der(A)$ we define a linear map $\Delta : Der(A) \to Der(A) \otimes Der(A)$, by $\Delta(D) = D \otimes 1 + 1 \otimes D$. Then, the following diagram commutes

$$\begin{array}{lll} A \otimes A & \stackrel{\Delta(D)}{\to} & A \otimes A, \\ \nu \downarrow & & \downarrow \nu \\ A & \stackrel{D}{\to} & A, \end{array} \tag{69}$$

where ν is the natural morphism defined by

$$\nu : \begin{array}{lll} A \otimes A & \to A, & \\ f \otimes g & \mapsto & f.g. \end{array} \tag{70}$$

We can extend Δ such that for each $\Delta(D_1D_2) = \Delta(D_1)\Delta(D_2)$, D_1, $D_2 \in Der(A)$, and the usual product on $\mathbb{R}\langle\langle Der(A)\rangle\rangle \otimes \mathbb{R}\langle\langle Der(A)\rangle\rangle$, $(a \otimes b).(c \otimes d) = (ac \otimes bd)$. For each word, $D_1 \dots D_r$, we define $\Delta(D_1 \dots D_r) = \Delta(D_1)\Delta(D_2 \dots D_r)$. We then extend Δ to $\mathbb{R}\langle\langle Der(A)\rangle\rangle$ by linearity. With this definition of Δ, for each $S \in \mathbb{R}\langle\langle Der(A)\rangle\rangle$, the following diagram commutes

$$\begin{array}{lll} A \otimes A & \stackrel{\Delta(S)}{\to} & A \otimes A, \\ \nu \downarrow & & \downarrow \nu \\ A & \stackrel{S}{\to} & A, \end{array} \tag{71}$$

As a consequence, we have the following lemma :

Lemma 4.5. — *The triple* $(\mathbb{R}\langle\langle Der(A)\rangle\rangle, \Delta, u)$ *is a bialgebra.*

4.2. Quantum bialgebra. — We follow the previous section on derivations.

Definition 4.1. — *For all* $\epsilon > 0$*, and* $\sigma = \pm$*, we denote by* $\bigtriangledown_\sigma^\epsilon$ *an operator acting on* $C^0 \otimes C^0$*, where* $\otimes$ *is the classical tensor product, and defined by*

$$\bigtriangledown_\sigma^\epsilon : \begin{array}{lll} C^0 \otimes C^0 & \to & C^0 \otimes C^0, \\ f \otimes g & \mapsto & \bigtriangledown_\sigma^\epsilon f \otimes g + f \otimes \bigtriangledown_\sigma^\epsilon g + \sigma\epsilon \bigtriangledown_\sigma^\epsilon f \otimes \bigtriangledown_\sigma^\epsilon g. \end{array} \tag{72}$$

Let $\Omega_\epsilon = \{\bigtriangledown_+^\epsilon, \bigtriangledown_-^\epsilon\}$ be the alphabet of two letters $\bigtriangledown_+^\epsilon$ and $\bigtriangledown_-^\epsilon\}$. We denote by Ω_ϵ^* the set of words $\underline{\omega} = \omega_1 \dots \omega_n$, $\omega_i \in \Omega_\epsilon$ for all $i = 1, \dots, n$, where $\omega_i\omega_j$ denote the natural composition of operators. For example, a possible word is $\bigtriangledown_+^\epsilon \bigtriangledown_-^\epsilon$.

Remark 4.1. — *Here we consider the alphabet of quantum operators for a fixed* $\epsilon > 0$*. It is possible that some particular problems of scale relativity require a complete alphabet* $\Omega = \{\bigtriangledown_\sigma^\epsilon, \sigma = \pm, \epsilon > 0\}$.

We denote by $A_\epsilon = \mathbb{R}\langle\langle \Omega_\epsilon \rangle\rangle$ the algebra of formal power series constructed on Ω_ϵ (with its classical algebraic structure).

We can define a linear map from A_ϵ to $A_\epsilon \times_{\mathbb{R}} A_\epsilon$, denoted Δ. First, we define Δ on $\bigtriangledown_\sigma^\epsilon$ by

$$\begin{array}{lll} A_\epsilon & \stackrel{\Delta}{\to} & A_\epsilon \otimes A_\epsilon, \\ \bigtriangledown_\sigma^\epsilon & \mapsto & \bigtriangledown_\sigma^\epsilon \otimes I + I \otimes \bigtriangledown_\sigma^\epsilon + \sigma\epsilon \bigtriangledown_\sigma^\epsilon \otimes \bigtriangledown_\sigma^\epsilon . \end{array} \tag{73}$$

We have the following equality

$$\bigtriangledown_\sigma^\epsilon \circ \nu = \nu \circ \Delta(\bigtriangledown_\sigma^\epsilon), \tag{74}$$

which is equivalent to the commutativity of the diagramm

$$\begin{array}{lll} C^0 \otimes C^0 & \stackrel{\Delta(\bigtriangledown_\sigma^\epsilon)}{\to} & C^0 \otimes C^0, \\ \downarrow \nu & & \downarrow \nu \\ C^0 & \stackrel{\bigtriangledown_\sigma^\epsilon}{\to} & C^0. \end{array} \tag{75}$$

We also define $\Delta(I) = I \otimes I$ in order that $I \circ \nu = \nu \circ \Delta(I)$.

We extend Δ to A_ϵ by linearity.

Lemma 4.6. — *The linear map Δ is an algebra homomorphism.*

Proof. The proof is done by induction. Let $\bigtriangledown^\epsilon_\sigma$ and $\bigtriangledown^\epsilon_{\sigma'}$ be two letters of Ω_ϵ. We have

$$\begin{aligned} \bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'}(fg) &= \bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} f.g + f.\bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} g \\ &+ \bigtriangledown^\epsilon_\sigma f.\bigtriangledown^\epsilon_{\sigma'} g + \bigtriangledown^\epsilon_{\sigma'} f.\bigtriangledown^\epsilon_\sigma g \\ &+\epsilon[\bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} f.\bigtriangledown^\epsilon_{\sigma'} g + \bigtriangledown^\epsilon_{\sigma'} f.\bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} g \\ &+ \bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} f.\bigtriangledown^\epsilon_\sigma g + \bigtriangledown^\epsilon_\sigma f.\bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} g] \\ &+\epsilon^2 \bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} f.\bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} g. \end{aligned} \tag{76}$$

As a consequence, we have

$$\begin{aligned} \Delta(\bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'}) &= \bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} \otimes I + I \otimes \bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} \\ &+ \bigtriangledown^\epsilon_\sigma \otimes \bigtriangledown^\epsilon_{\sigma'} + \bigtriangledown^\epsilon_{\sigma'} \otimes \bigtriangledown^\epsilon_\sigma \\ &+\epsilon[\bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} \otimes \bigtriangledown^\epsilon_{\sigma'} + \bigtriangledown^\epsilon_{\sigma'} \otimes \bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} \\ &+ \bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} \otimes \bigtriangledown^\epsilon_\sigma + \bigtriangledown^\epsilon_\sigma \otimes \bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'}] \\ &+\epsilon^2 \bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} \otimes \bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'} . \end{aligned} \tag{77}$$

By definition, the classical product on $A_\epsilon \otimes A_\epsilon$ is defined by : for all $a \otimes b$ and $c \otimes d$ in $A_\epsilon \otimes A_\epsilon$, we have

$$(a \otimes b).(c \otimes d) = ac \otimes bd. \tag{78}$$

An easy computation proves that

$$\Delta(\bigtriangledown^\epsilon_\sigma \bigtriangledown^\epsilon_{\sigma'}) = \Delta(\bigtriangledown^\epsilon_\sigma)\Delta(\bigtriangledown^\epsilon_{\sigma'}). \tag{79}$$

By induction and linearity, we obtain the lemma. □

We define an algebra homomorphism from A_ϵ to $\mathbb{R}$, denoted by u, by associating to each formal power series its constant term.

With $u : A_\epsilon \to \mathbb{R}$ and $\Delta : A_\epsilon \to A_\epsilon \otimes A_\epsilon$, we define a coalgebra structure on A_ϵ.

Lemma 4.7. — *The triple (A_ϵ, Δ, u) is a coalgebra.*

Moreover, as u and Δ are homomorphism, we obtain the stronger result :

Lemma 4.8. — *The triple (A_ϵ, Δ, u) is a bialgebra.*

We have not found a natural graduation on this bialgebra.

Remark 4.2. — *1. It will be interseting to discuss the possible relation to Quantum groups introduced by Drinfeld ([**22**],[**23**]) and quasi-triangular Hopf algebra. We introduce the natural commutativity involution*

$$\tau : \begin{array}{ccc} A_\epsilon \otimes A_\epsilon & \to & A_\epsilon \otimes A_\epsilon, \\ a \otimes b & \longmapsto & b \otimes a. \end{array} \tag{80}$$

Let $\Delta^{op} = \tau \circ \Delta$. The map τ is an algebra automorphism of $A_\epsilon \otimes A_\epsilon$ and the following diagram commutes

$$\begin{array}{ccc} A_\epsilon & \xrightarrow{\Delta} & A_\epsilon \otimes A_\epsilon, \\ id_{A_\epsilon} \downarrow & & \downarrow \tau \\ A_\epsilon & \xrightarrow{\Delta^{op}} & A_\epsilon \otimes A_\epsilon. \end{array} \tag{81}$$

*If (A_ϵ, Δ, u) is a quasitriangular algebra, then, following [**43**], there exists an invertible element of $A_\epsilon \otimes A_\epsilon$ such that*

$$\Delta^{op}(a) = R\Delta(a)R^{-1}, \text{ for all } a \in A_\epsilon. \tag{82}$$

In our case, we easily have $\Delta^{op} = \Delta$ so that (A_ϵ, Δ, u) so that R is trivial.

2. A natural idea is to consider the quantum bialgebra as a "deformation" of the classical Hopf algebra associated to derivations. The word derivation must be taken with care, because a notion of deformation for Hopf algebra already exists and it is not clear if this is the good one to consider here.

3. There is no natural extension of the quantum bialgebra in order to take into account the scale derivative. The basic problem being that the scale derivative is a complex valued operator on real valued functions. Then, we have no natural composition of these operators.

4.3. A remark on Rieman-Liouville fractional calculus and the local fractional calculus. — A basic way to deal with non differentiable functions

is to use *fractional calculus*. As an example, one can consider the classical (left and right) *Riemann-Liouville derivative*, defined by

$$D_z^\alpha f(x) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_0^{z^+} f(t)(t-z)^{-\alpha-1}dt, \tag{83}$$

for $\alpha \in \mathbb{C} \setminus \{-1,-2,\dots\}$. The Riemann-Liouville derivative is not at all a *derivation* on the algebra of continuous functions. Indeed, one has (see [**40**]) :

$$D_z^\alpha(uv) = \sum_{n=0}^{\infty} \begin{pmatrix} \alpha \\ n \end{pmatrix} D_z^{\alpha-n} u D_z^n v. \tag{84}$$

Moreover, this operator, which is a direct analytic generalization of Cauchy formula doesn't have a clear geometrical interpretation, despite recent advances (see [**3**]).

In [**5**], we have obtain, following a previous work of Kolvankar and Gangal [**29**], a derivation, called the (left and right) *local fractional derivative*, by *localyzing* the (left and right) Riemann-Liouville derivative.

We have prove that this localization takes a simple form.

Definition 4.2. — *The right and left local fractional derivative of f at point x_0 of order α is defined by*

$$\lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{(x-x_0)^\alpha}, \text{ and } \lim_{x\to x_0^-} \frac{f(x_0)-f(x)}{(x_0-x)^\alpha}, \tag{85}$$

respectively.

We have introduce the fractional derivative of f at point x_0 by collecting the two quantities $d_+^\alpha f(x_0)$ and $d_-^\alpha f(x_0)$ in a single quantity, i.e.

$$d^\alpha f(x_0) = \frac{1}{2}\left(d_+^\alpha f(x_0) + d_-^\alpha f(x_0)\right) + i\frac{1}{2}\left(d_+^\alpha f(x_0) + d_-^\alpha f(x_0)\right). \tag{86}$$

Moreover, we obtain a clear geometrical meaning by connecting the exponent α of differentiation to the local maximal Hölder regularity of the curve.

However, such a derivative has important problems. First, there exist no *integral operator* contrary to the Riemann-Liouville fractional derivative where

there exists the Riemann-Liouville integral. Moreover, the set of points on which the local fractional derivative is non zero is in most of the case trivial, i.e. of zero measure. Precisely, we have the following theorem.

Theorem 4.1. — *The fractional differential equations of the form* $d^\alpha f(x) = a(x) + ib(x)$, $0 < \alpha < 1$, *where* $a(x)$ *and* $b(x)$ *are continuous functions such that there exists* $x_0 \in \mathbb{R}$ *such that* $\mid a(x_0) \mid \neq \mid b(x_0) \mid$, *have no solutions.*

This theorem solves a conjecture of [**7**]. It must be pointed out that the condition $\mid a(x_0) \mid \neq \mid b(x_0) \mid$ is generic.

Proof. We have $d^\alpha_+ f(x) = c(x)$ and $d^\alpha_- f(x) = d(x)$, where $c(x) = a(x)+b(x)$ and $d(x) = a(x) - b(x)$ are continuous functions. By assumption, we have $c(x_0) \neq 0$ and $d(x_0) \neq 0$.

Let us assume that $c(x_0) > 0$ (the case where $c(x_0) < 0$ is similar). By continuity, there exits $\epsilon > 0$ such that for all x in the open interval $I_\epsilon(x_0) =]x_0 - \epsilon, x_0 + \epsilon[$, we have $d^\alpha f(x) > 0$. Two case must be considered :

i) $d^\alpha_- f(x_0) > 0$ and ii) $d^\alpha_- f(x_0) < 0$.

In the case i), we define an open interval $J_\epsilon(x_0)$ such that for all $x \in J_\epsilon(x_0)$ we have $d^\alpha f(x) > 0$. As a consequence, the function f is Hölderian of exponent α (see [**4**],theorem 3.9). Moreover, the function f is injective on the interval $x \in K_\epsilon(x_0) = J_\epsilon(x_0) \bigcap I_\epsilon(x_0)$. Indeed, if there exists $x_1, x_2 \in K_\epsilon(x_0)$ such that $f(x_1) = f(x_2)$, then by the fractional Rolle's theorem ([**4**]), there $x \in K_\epsilon(x_0)$ such that $d^\alpha_+ f(x) d^\alpha_- f(x) \leq 0$, which is impossible by assumption. A continuous function wich is injective is strictly monotone (see [**28**],lemma 3.8,p.207). But, a monotone function is almost everywhere differentiable by Lebesgue theorem ([**31**],p.319), in contradiction with the assumption that $0 < \alpha < 1$.

In the case ii), we define an open interval $J_\epsilon(x_0)$ such that for all $x \in J_\epsilon(x_0)$ we have $d^\alpha f(x) < 0$. Again the function f is an Hölderian function of exponent α. Moreover, by ([**4**],theorem 4.9), all points in $K_\epsilon(x_0) = J_\epsilon(x_0) \bigcap I_\epsilon(x_0)$ are local minima. This is possible if and only if f is a constant function, i.e. a differentiable function in contradiction with the fact that f is Hölderian of exponent $0 < \alpha < 1$. This concludes the proof of the theorem. □

We can characterize the *spectrum* of f, i.e. the set of values taken by the fractional derivative of a continuous function.

Corollary 4.1. — *The spectrum of a given α-differentiable function is discontinuous or zero.*

Proof. The spectrum can't be continuous and non zero by theorem 4.1. As a consequence, the spectrum can be zero or dicontinuous. □

We can be more precise on the nature of this spectrum.

Theorem 4.2. — *The set of fractional differential equations $d^\alpha f(x) = a(x) + ib(x)$, where $0 < \alpha < 1$ and the functions $c(x) = a(x)+b(x)$, $d(x) = a(x)-b(x)$ keep a constant sign on a given interval of $\mathbb{R}$, has no solutions.*

Proof. This is the same proof as theorem 4.1. The continuity assumption of theorem 4.1 being only here to construct an interval where $c(x)$ and $d(x)$ keep a constant sign. □

As a consequence, the spectrum is discontinuous but of very special form, as we can't find any interval of $\mathbb{R}$ on which we have a constant sign. We define the *Dirichlet function* as follows

$$\mathcal{D}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \tag{87}$$

Then, the function $a(x)$ and $b(x)$ can be taken such that

$$a(x) = \frac{3}{2}\mathcal{D}(x), \quad b(x) = -\frac{1}{2}\mathcal{D}(x). \tag{88}$$

We can extend all the previous theorem to the case where the order of fractional differentiation is non constant, but a function of x.

In [**5**], we have derived , in the fractional calulus framework, the Schrödinger equation from the Newton's equation of dynamics under the assumption that one-dimensional quantum mechanical trajectories satisfy

$$(d_\sigma^\alpha f(x))^2 = 2\bar{h}/2m,\ \sigma = \pm, \tag{89}$$

where $\bar{h}$ is the reduced Planck constant $h/2\pi$ and m is the mass of the particle. Of course, this is impossible by theorem 4.1, as already proved in [**7**]. But as the Schrödinger equation is a well established equation of physics, we propose in [**7**] to consider a small perturbation of condition (89), like

$$(d_\sigma^\alpha f(x))^2 = 2\bar{h}/2m + \epsilon a_\sigma(t, f(t)),\ \sigma = \pm \tag{90}$$

where $0 < \epsilon << 1$, in order to permit the existence of non trivial solutions and to obtain a small perturbation of the Schrödinger equation.

However, condition (90) lead to a deadlock. Indeed, for ϵ sufficiently small, the quantities $d_\sigma^\alpha f(x)$ keep a constant sign. As a consequence, by theorem 4.2 we have no solutions.

All these problems are solved in part II by using the scale calculus framework.

5. Quantum representation of non differentiable functions

We introduce the notion of quantum geometric representation for a continuous non differentiable function. This notion is associated to minimal resolution and the scale derivative. It turns out that a geometric space displaying the basic features of the quantum geometric representation is given by a simplified version of A. Connes formulation of the standard model of fundamental interactions within the framework of noncommutative geometry.

5.1. The quantum representation of a continuous non differentiable function. — Our previous results allow us to define a natural notion of scale derivative. The scale derivative, which is a complex valued operator, contains the necessary informations in order to perform a local analysis of a continuous non differentiable functions, and take care of this non differentiability.

Let f be a given continuous on differentiable function. The basic functions associated to f, and from which we can deduce the scale derivative are the forward and backward mean functions defined as $f_\epsilon^+(t) = (1/2\epsilon)\int_t^{t+\epsilon} f(s)ds$ and $f_\epsilon^-(t) = (1/2\epsilon)\int_{t-\epsilon}^{t} f(s)ds$ respectively.

From the geometrical view point, it means that in order to take into account the non differentiable character of f, one must consider the disjoint union $\Gamma_\epsilon^+ \cup \Gamma_\epsilon^-$, where Γ^σ is the graph of f_ϵ^σ, $\sigma = \pm$.

Definition 5.1. — *Let $h > 0$, f be a continuous non differentiable function, and $\epsilon(f,h)$ be its minimal resolution. For all $\epsilon > 0$, the quantum geometric representation of f, denoted $Q_\epsilon(f)$, is defined by :*

i) For all $\epsilon > \epsilon(f,h)$, $Q_\epsilon(f) = \Gamma_\epsilon$,

ii) For all $0 < \epsilon \leq \epsilon(f,h)$, $Q_\epsilon(f) = \Gamma_\epsilon^+ \cup \Gamma_\epsilon^-$,

where Γ_ϵ, Γ_ϵ^+, Γ_ϵ^-, are the graph of the mean function f_ϵ, the forward mean function and the backward mean function respectively.

The non differentiability of f induces a change in the geometric structure of the geometric representation of f.

In the following, we consider graphs of real valued functions as submanifold of $\mathbb{R}^2$. As f_ϵ^+ and f_ϵ^- are differentiable functions, one deals with a disjoint union of differentiable submanifolds Γ_ϵ^+ and Γ_ϵ^-. The basic features of the

quantum representation of f, when $0 < \epsilon < \epsilon(f,h)$, is that Q_ϵ is composed by two differentiable submanifolds, which are close to each other, their closeness being related to ϵ.

As a consequence, a good understanding of the effects of a non differentiable function can be obtained via the following simplified model :

Let M be a one dimensional differentiable submanifold of $\mathbb{R}^2$. Let $A = \{a, b\}$ be a two points space. We consider $Q = M \times A$. Then, Q is the union of two copies of the manifold M : $Q = M_a \cup M_b$.

Remark 5.1. — *A more accurate model is the following : Let $A_\epsilon = \{a_\epsilon, b_\epsilon\}$ be a two points space, such that for $\epsilon > \epsilon(f,h)$, A_ϵ reduces to a point, i.e. $a_\epsilon = b_\epsilon$. The simplified model is then $Q_\epsilon = M \times A_\epsilon$.*

5.2. Noncommutative geometry. — In this paragraph we only sketch a possible connexion between our point of view on non differentiable functions and non commutative geometry. The idea is, by this way, to obtain powerfull tools to study non differentiable functions which will be relevant to physics.

In his book [**15**], A. Connes developp non commutative geometry. The basic idea is to extend to the non commutative case the classical result of Gelfand and Naimark relating C^*-algebras and locally compact spaces.

At the end of his book, Connes ([**15**],p.568) discusses a particular example, where his theory already lead to interesting results (this example is view as a preliminary step toward a complete non commutative model for the standard model of quantum particles).

He considers a product of a differentiable manifold (the standard $\mathbb{R}^4$) by a discrete space, $A = \{a, b\}$. Using non commutative geometry, he can make analysis on this space. In particular, he defines a "differential" operator which

contains three terms : the classical derivative on each copie of $\mathbb{R}^4$ and a finite difference.

It will be interesting to discuss the relevance of this construction with respect to our approach to non differentiable functions.

PART II

SCALE RELATIVITY AND THE SCHRÖDINGER EQUATION

1. Introduction

The Schrödinger equation is one of the basic pieces of quantum mechanics. Many attempts already exist in order to derive it from the expected behaviours of trajectories of quantum particles or from classical equations of the dynamics. We can cite for example :

- Nelson stochastic approach [**36**],

- Feynman perturbativ approach [**25**] ,

- Nottale's approach by the Scale relativity theory [**37**] .

In the following we discuss the derivation of the Schrödinger equation in the framework of the Scale relativity of Nottale [**37**]. The main point is that, contrary to Nelson or Feynman approach, it is based on a *first principle*, namely, the *scale principle of relativity*, which is an extension of the Einstein relativity principle to scales (of time and lengths).

The scale relativity principle introduced by Nottale's has a direct consequence on the equations of the dynamics for a given particle. Indeed, they must keep the same form under a scale transform, i.e. going from the classical scale to the atomic scale. Following Feynman and Hibbs, the principal difference between the microscopic and macrocospic scale is that typical paths become non differentiable. Then, we must be able to transform the classical differential equations of the dynamics for functions which are not at all differentiable.

This is done using the scale difference operator defined in part I. The scale relativity principle is then equivalent to changing the classical derivative by the scale difference operator in the Euler-Lagrange equations of the dynamics.

This quantification procedure called the scale quantization can be precisely defined in §.2, by introducing a quantization map, associating to each classical variables and differential operators its quantum counterpart. One of the main problem is then that the scale quantization procedure of the Euler-Lagrange equation is not unique. Indeed, we can first quantify the Lagrangian of the system and then define a quantized Euler-Lagrange equation, or wa can quantify directly the classical Euler-Lagrange equation. The main point, proved in the coherence lemma in §.4.2, is that these two procedures coincide.

The scale quantization procedure being precisely defined, we can specialise it to the quantum mechanical case. The principal free parameter in the quantization lies in the order of the regularity of the non differentiable curve, i.e. its Hölder exponent. Using the Feynman-Hibbs characterization of quantum paths, as well as the Heisenberg inequalities, we prove in §.3 that the Hölder regularity of a quantum path is $1/2$. Using this result, we prove in §.4.4 that the quantized analogue of the Newtonian equation of dynamics is a generalized non linear Schrödinger equation. This is done by introducing a wave-function in §.4.3, which is the direct consequence of the complex nature of the speed, being itself a consequence of the non differentiability of the curve. Under special

assumptions, which can be interpreted, we recover the classical Schrödinger equation.

2. Scale quantization procedure for classical Lagrangian systems

2.1. Classical Lagrangian systems and Euler-Lagrange equation. — In this article, we only discuss classical Lagrangian systems defined as follow :

Definition 2.1. — *A Lagrangian $L(x,v,t)$ is called classical if it is of the form kinetic energy+potential, i.e.*

$$L(x,v,t) = K(v) + U(x,t), \tag{1}$$

where $K(v)$ is a quadratic form.

The basic example for $K(v)$ is the classical kinetic energy of a particule of mass m given by

$$K(v) = \frac{1}{2}mv^2. \tag{2}$$

The dynamics associated to a Lagrangian system is determined by the *Euler-Lagrange equations*.

Definition 2.2. — *Let $L(x,v,t)$ be a classical Lagrangian system. The Euler-Lagrange equation associated to L is the following partial differential equation :*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(x(t),v(t),t)\right) = \frac{\partial L}{\partial x}(x(t),v(t),t). \tag{EU}$$

We denote by E the mapping associating to L its Euler-Lagrange equation (EU).

2.2. The scale quantization procedure. — In this section, we define the *scale quantization procedure*, which formalizes Nottale's approach to quantum mechanics. The terminology suggests that the quantization procedure follows ideas coming from the theory of the *scale relativity* developped by Nottale [**37**].

2.2.1. The scale quantization map. — We define a map Q which acts on differential operators, variables and functions.

The classical variables x, v, t have quantized analogues which are denoted by $X = Q(x)$, $V = Q(v)$ and $T = Q(t)$.

In the following, we assume that :

Assumption 2.1. — *We have* $Q(t) = t$.

The time variable has then a specific role, being the only variable not affected by the quantization procedure.

We denote also by

$$X(t) = Q(x(t)),\ V(t) = Q(v(t)), \tag{3}$$

the quantized version of the position trajectory and speed.

The main point is that we don't know for the moment the regularity of $X(t)$ or $V(t)$.

The first algebraic properties of Q is the following :

***Property 2.1* (Quantization of maps)**. — *We consider a map* $L : (x, v, t) \mapsto L(x, v, t)$. *The quantized map* $Q(L) = \mathcal{L}$ *is defined by*

$$\mathcal{L} : (X, V, t) \mapsto L(X, V, t). \tag{4}$$

As a consequence, if L is differentiable with respect to the variable x, v or t then $\mathcal{L}$ is differentiable with respect to X, V or t.

In order to use Q on differential equations, we must precise its behaviour with respect to differential operators.

***Property 2.2* (Operator).** — *We consider a map of the form* $f(t) = L(x(t), v(t), t)$*, where* $x(t)$ *and* $v(t)$ *are differentiable functions. The differential operator* d/dt *acts on operator* f*. By the map* Q*, we define a quantized operator* $Q(d/dt)$ *such that*

i) $Q(df/dt) = Q(d/dt).Q(f)$*,*

where $Q(d/dt)$ *is on operator acting on* $Q(f)$*, depending on the regularity of* $Q(f)$ *with respect to* t *:*

**) If* $Q(f)(t)$ *is differentiable with respect to* t*, then* $Q(d/dt) = d/dt$*.*

***) If* $Q(f)(t)$ *is non differentiable with respect to* t*, then* $Q(d/dt) = \square_\epsilon/\square t$*, where* $\epsilon(X, h) > \epsilon > 0$*,* h *being a constant.*

The constant h must be fixed by physical constraint. In the following, we consider h has a free parameter.

As $f(t) = L(x(t), v(t), t)$, we have $Q(f)(t) = L(X(t), V(t), t)$. Hence, the regularity of $Q(f)$ with respect to t depends on the regularity of $X(t)$ and $V(t)$ with respect to t.

Moreover, as $v = dx/dt$, we have

$$V(t) = Q\left(\frac{d}{dt}\right)[X(t)]. \tag{5}$$

Hence, the regularity of $X(t)$ can induce a change in the form of the speed $V(t)$.

2.2.2. Scale quantization of the Euler-Lagrange equation. — By the quantization procedure we give the quantized version of the Euler-Lagrange formula (7).

Lemma 2.1. — *The quantized Euler-Lagrange equation $Q(EU)$ is given by*

$$Q\left(\frac{d}{dt}\right)\left[\frac{\partial \mathcal{L}}{\partial V}(X(t),V(t),t)\right]=\frac{\partial \mathcal{L}}{\partial X}. \tag{6}$$

Proof. The action of Q on the classical Euler-Lagrange equation (EU) gives

$$Q\left(\frac{d}{dt}\right)\left[Q\left(\frac{\partial L}{\partial v}(x(t),v(t),t)\right)\right]=Q\left(\frac{\partial L}{\partial x}\right). \tag{7}$$

As L is assumed to be differentiable with respect to the variables v and x, we have, using property 2.1 :

$$Q\left(\frac{d}{dt}\right)\left[\frac{\partial Q(L)}{\partial Q(v)}(Q(x)(t),Q(v)(t),t)\right]=\frac{\partial Q(L)}{\partial Q(x)}, \tag{8}$$

With our notations, equation (8) gives equation (6). This concludes the proof of the lemma. □

As a consequence, in order to precise the quantization procedure, we only have to precise the regularity of $Q(x(t))=X(t)$.

3. Generic trajectories of Quantum mechanics

In order to precise the quantization procedure, we investigate the regularity of quantum-mechanical path.

3.1. Feynman and Hibbs genericity condition. — Feynman and Hibbs have already noted in ([**26**],p.176-177) that typical path of quantum-mechanical particle is continuous and nondifferentiable. More precisely, there exist a quadratic velocity, i.e. if $X(t)$ denotes the particle trajectory, then

$$\lim_{t\to t'}\frac{(X(t)-X(t'))^2}{t-t'}\ \text{exists.} \tag{FH}$$

As a consequence, we have the following result.

Lemma 3.1. — *Under Feynman-Hibbs characterization (FH), we have* $X(t)\in H^{1/2}$.

We can deduce the following result on the Hausdorff dimension of typical paths of quantum mechanics :

Corollary 3.1. — *Under Feyman-Hibbs characterization (FH), the Hausdorff dimension of X is* $1/2$.

Proof. As $X \in H^{1/2}$, this follows from (theorem 20.6,[**46**],p.310). □

In the contrary, the fractal (or Minkowski-Bouligand) dimension Δ is given by

$$\Delta(X) = 2 - (1/2) = 3/2, \tag{9}$$

using ([**46**],p.154-155).

This result has been discussed in great details by Abbott and Wise [**1**].

3.2. Heisenberg uncertainty principle. — The non-differentiable character of typical paths of quantum mechanics can be seen as a consequence of *Heisenberg uncertainty relations*. We refer to ([**37**],p.93-95) and ([**14**],p.130-131) for details.

Let Δx, Δt and Δp be the precision of the measurement of the position x, time t and *momentum* p of a given particle. The Heisenberg uncertainty relation on momentum and position is

$$\Delta p \Delta x \geq h. \tag{10}$$

We have the following relations

$$\begin{array}{rcl} x(t \pm \Delta t) & = & x(t) \pm \Delta x, \\ v_\epsilon(t \pm \Delta t) & = & (x(t+\epsilon) - x(t)/\epsilon) \pm \Delta v, \end{array} \tag{11}$$

where $\epsilon \geq \Delta t$ by definition of Δt, and

$$\Delta p = m \Delta v. \tag{12}$$

Remark 3.1. — *The speed of a particle $v(t)$ is defined as the limit of the mean-speed $v_\epsilon(t) = (x(t+\epsilon) - x(t)/\epsilon)$ when ϵ goes to zero. As the time variable t is known only with precision Δt, we must have $\epsilon \geq \Delta t$.*

What are the relations between Δv, Δx and Δt ?

As $\epsilon \geq \Delta t$ by assumption, we have by taking the best possible value $\epsilon = \Delta t$,

$$\Delta v \sim 2\Delta x/\Delta t. \tag{13}$$

As a consequence, the Heisenberg uncertainty relation (10) gives

$$m\frac{(\Delta x)^2}{\Delta t} \sim h. \tag{14}$$

We deduce that

$$\Delta x \sim \frac{h}{2m}\Delta t^{1/2}. \tag{15}$$

Hence, we deduce that Heisenberg uncertainty relation (10) induce the fact that $X \in H^{1/2}$.

Of course, the previous reasoning can be reversed. If typical paths of quantum mechanics are assumed to be in $H^{1/2}$ then we obtain Heisenberg-like uncertainty relations.

Lemma 3.2. — *Let $X \in H^{1/2}$, and $0 < \Delta t << 1$ be a small parameter. We denote by $\mathcal{X}$ the graph of X, and by $\mathcal{X}(t) = (t, X(t))$ a point belonging to $\mathcal{X} \in \mathbb{R}^2$. We denote by $\Delta x(t) = \| \mathcal{X}(t), \mathcal{X}(t+\Delta t) \|$, where $\| \, . \, \|$ is the classical Euclidean norm on $\mathbb{R}^2$. We have*

$$\Delta x \sim (\Delta t)^{1/2}. \tag{16}$$

for all $t \in \mathbb{R}$.

Proof. This follows from a simple computation. We have

$$\| \mathcal{X}(t), \mathcal{X}(t+\Delta t) \|^2 = (\Delta t)^2 + (X(t+\Delta t) - X(t))^2. \tag{17}$$

As $X \in H^{1/2}$, we obtain

$$\| \mathcal{X}(t), \mathcal{X}(t+\Delta t) \|^2 \leq (\Delta t)^2 + C^2\Delta t \sim \Delta t, \tag{18}$$

for Δt sufficiently small. □

Of course, this result extends to arbitrary $X \in H^\alpha$, with $0 < \alpha < 1$, for which we obtain

$$\Delta x \sim (\Delta t)^\alpha. \tag{19}$$

4. Scale quantization of Newtonian mechanics

4.1. Quantization of speed. — The fact that typical path of quantum mechanics belongs to $H^{1/2}$ has many implications with respect to the quantization procedure. The first one being that the quantized version of the speed v is now complex.

***Lemma 4.1* (Quantized speed).** — *Let $Q(x) = X \in H^{1/2}$, then $Q(v) = V \in \mathbb{C}$. Precisely, we have*

$$V = \frac{\Box_\epsilon X}{\Box t}. \tag{20}$$

Proof. This follows from the quantization procedure. We have

$$v = \frac{dx}{dt}. \tag{21}$$

By Q, we obtain

$$V = Q\left(\frac{d}{dt}\right)(X). \tag{22}$$

As $X \in H^{1/2}$, we have

$$Q(d/dt) = \Box_\epsilon / \Box t, \tag{23}$$

which is a complex number by definition. □

4.2. Scale relativity and the coherence lemma. —

4.2.1. Scale relativity and Scale Euler-Lagrange equations. — The scale-relativity theory developped by Nottale ([**37**],[**38**]) extends the Einstein relativity principle to *scale*. An heuristic version of the new relativity principle can be written as :

"*The equations of physics keep the same form under any transformation of scale (contractions and dilatations).*"

The mathematical foundation of such a theory is difficult. The main difficulty being that space-time is now a non differentiable manifold. We refer to the work of Nottale [**37**] for more details.

The scale relativity principle has a direct consequence on the form of the equation of motion for a particle.

We first introduce the following "Scale" Euler-Lagrange equation :

Definition 4.1. — *Let $\mathcal{L}(V,t)$ be a quantized Lagrangian. The Scale Euler-Lagrange equation associated to $\mathcal{L}(X,V,t)$ is the equation*

$$\frac{\square_\epsilon}{\square t}\left(\frac{\partial \mathcal{L}}{\partial V}(X(t),V(t),t)\right) = \frac{\partial \mathcal{L}}{\partial X}(X(t),V(t),t). \qquad (\mathcal{E}U)$$

We denote by $\mathcal{E}$ the mapping associating equation $(\mathcal{E}U)$ to $\mathcal{L}$.

The scale relativity principle is then equivalent to the following statement :

***Statement 4.1* (Scale relativity)**. — *The equation of motion of a quantum-mechanical particle satisfy the Scale Euler-Lagrange equation.*

The main point is that the Scale Euler-Lagrange equation doesn't follow from the quantization procedure, but from a first principle which fixes the form of the equation of motions.

4.2.2. Coherence lemma. — At this point, we can state the *coherence lemma*, which ensures us that the quantization procedure is well defined. Indeed, the scale relativity principle gives an equation of motion particle in quantum mechanics, i.e. $(\mathcal{E}U)$. But, the quantization procedure can be used to obtain an equation of motion for the particle as a quantized version of the classical Euler-Lagrange equation, i.e. $Q(EU)$. The main point is that this two constructions are equivalent, i.e. $Q(EU) = \mathcal{E}U$. Precisely, we have :

***Lemma 4.2* (Coherence).** — *The following diagramm commutes*

$$\begin{array}{ccc} L(x,v,t) & \xrightarrow{Q} & \mathcal{L}(X,V,t) \\ E\downarrow & & \downarrow\mathcal{E} \\ \dfrac{d}{dt}\left[\dfrac{\partial L}{\partial v}(x(t),v(t),t)\right]=\dfrac{\partial U}{\partial x} & \xrightarrow{Q} & \dfrac{\square_\epsilon}{\square t}\left[\dfrac{\partial \mathcal{L}}{\partial V}(X(t),V(t),t)\right]=\dfrac{\partial U}{\partial X}. \end{array} \tag{24}$$

Proof. This follows from a direct computation. □

In term of mapping, the coherence lemma is then equivalent to

$$Q\circ E=\mathcal{E}\circ Q. \tag{25}$$

4.3. Action functional and wave. — A basic element of Lagrangian mechanics is the action functional $A(x,t)$ which is related to speed via the equation

$$v=\frac{1}{m}\frac{\partial A}{\partial x}. \tag{26}$$

The function $A(x,t)$ is differentiable with respect to x and t. By the quantization procedure, we obtain the analogue of the classical action functional for quantum mechanics :

***Lemma 4.3* (Quantized action functional).** — *The function $\mathcal{A}=Q(A)$ is a complex valued function $\mathcal{A}(X,t)$ which satisfies*

$$V=\frac{1}{m}\frac{\partial \mathcal{A}}{\partial X}. \tag{27}$$

Proof. By Q the classical equation (26) gives

$$V=Q(v)=Q\left(\frac{\partial}{\partial x}\right)Q(A). \tag{28}$$

As the classical action $A(x,t)$ is differentiable with respect to x, the quantized version $\mathcal{A}(X,t)$ is differentiable with respect to X. As a consequence, we have

$$Q\left(\frac{\partial}{\partial x}\right)=\frac{\partial}{\partial X}. \tag{29}$$

This concludes the proof. □

Moreover, the quantization procedure gives us the following relation :

***Lemma 4.4* (Action).** — *The quantized action $\mathcal{A}(x,t)$ satisfies*

$$\mathcal{L} = \frac{\partial \mathcal{A}}{\partial t}. \tag{30}$$

At this point, the main feature of quantum mechanics is to introduce complex speed and action functional. We can discuss the behaviour of $\mathcal{A}$ by introducing a complex valued function, which is called the *wave function*.

***Definition 4.2* (Wave function).** — *We called wave function associated to X, the complex valued function defined by*

$$\psi(X,t) = \exp\left(\frac{i\mathcal{A}(X,t)}{2m\gamma}\right), \tag{31}$$

where $\gamma \in \mathbb{R}$ is a constant number.

The constant γ is a normalization constant, which depends on the regularity property of $X(t)$.

The complex nature of V leads naturally to the introduction of the wave function. The wave formalism is then induced by the non differentiable character of typical paths of quantum mechanics.

4.4. The quantized Euler-Lagrange equation. — The quantized Euler-Lagrange equation is given by

$$m\frac{\Box_\epsilon V(t)}{\Box t} = \frac{dU}{dx}(x). \tag{32}$$

The complex speed V is related to the wave function, so that equation (32) can be writen in term of ψ.

***Theorem 4.1*.** — *Let $X(t)$ be a continuous non differentiable function in $H^{1/2}$ and ψ its associated wave function. The quantized Euler-Lagrange equation is of the form*

(33)

$$2i\gamma m\left[-\frac{1}{\psi}\left(\frac{\partial\psi}{\partial X}\right)^2\left(i\gamma + \frac{a_\epsilon(t)}{2}\right) + \frac{\partial\psi}{\partial t} + \frac{a_\epsilon(t)}{2}\frac{\partial^2\psi}{\partial X^2}\right] = (U(x)+\alpha(x))\psi + o(\epsilon^{1/2}),$$

where $\alpha(x)$ is an arbitrary continuous function, and

$$a_\epsilon(t) = \frac{1}{2}\left[(\Delta_\epsilon^+ X(t))^2 - (\Delta_\epsilon^- X(t))^2\right] - i\frac{1}{2}\left[(\Delta_\epsilon^+ X(t))^2 + (\Delta_\epsilon^- X(t))^2\right] .. \tag{34}$$

The equation (33) is called *generalized Schrodinger equation* by Nottale [**37**].

Proof. We have

$$V = -i2\gamma\frac{\partial \ln(\psi)}{\partial X}. \tag{35}$$

The Euler-Lagrange equation is now given by

$$2i\gamma m\frac{\Box_\epsilon}{\Box t}\left(\frac{\partial \ln(\psi)}{\partial X}\right) = \frac{dU}{dX}. \tag{36}$$

We denote

$$f(X,t) = \frac{\partial \ln(\psi(X,t))}{\partial X}(X,t). \tag{37}$$

We apply the main lemma of part I to compute $\Box_\epsilon f(X(t),t)/\Box t$ for complex valued functions. We have

$$\begin{aligned}\frac{\Box_\epsilon}{\Box t}\left(\frac{\partial \ln(\psi)}{\partial X}(X(t),t)\right) &= \frac{\Box_\epsilon X}{\Box t}\frac{\partial}{\partial X}\left(\frac{\partial \ln(\psi(X,t))}{\partial X}\right)(X(t),t)\\ &+\frac{\partial}{\partial t}\left(\frac{\partial \ln(\psi(X,t))}{\partial X}\right)(X(t),t)\\ &+\frac{1}{2}a_\epsilon(t)\frac{\partial^2}{\partial X^2}\left(\frac{\partial \ln(\psi(X,t))}{\partial X}\right)(X(t),t) + o(\epsilon^{1/2}).\end{aligned} \tag{38}$$

Elementary calculus gives

$$\frac{\partial \ln(\psi(X,t))}{\partial X} = \frac{1}{\psi}\frac{\partial \psi}{\partial X}, \quad \text{and} \quad \frac{\partial}{\partial X}\left(\frac{1}{\psi}\frac{\partial \psi}{\partial X}\right) = \frac{1}{\psi}\frac{\partial^2 \psi}{\partial^2 X} - \frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial X}\right)^2. \tag{39}$$

Moreover, by definition of the wave function ψ and V, we have

$$V = -i2\gamma\frac{\partial \ln \psi(X,t)}{\partial X}. \tag{40}$$

Hence, we obtain

$$\frac{\Box_\epsilon X}{\Box t} = V = -i2\gamma\frac{\partial \ln \psi(X,t)}{\partial X}, \tag{41}$$

and

$$\frac{\Box_\epsilon X}{\Box t}\frac{\partial}{\partial X}\left(\frac{\partial \ln(\psi(X,t))}{\partial X}\right)(X(t),t) = -i2\gamma\frac{\partial \ln(\psi)}{\partial X}\frac{\partial}{\partial X}\left(\frac{\partial \ln(\psi)}{\partial X}\right)(X(t),t), \tag{42}$$
$$= -i\gamma\frac{\partial}{\partial X}\left[\left(\frac{\partial \ln(\psi)}{\partial X}\right)^2\right](X(t),t),$$
$$= -i\gamma\frac{\partial}{\partial X}\left[\frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial X}\right)^2\right](X(t),t).$$

We then have

$$\frac{\Box_\epsilon}{\Box t}\left(\frac{\partial \ln(\psi(X,t))}{\partial X}(X(t),t)\right) \tag{43}$$
$$= \frac{\partial}{\partial X}\left[-i\gamma\frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial X}\right)^2 + \frac{\partial \ln(\psi)}{\partial t} + \frac{1}{2}a_\epsilon(t)\left[\frac{1}{\psi}\frac{\partial^2\psi}{\partial X^2} - \frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial X}\right)^2\right]\right] + o(\epsilon^{1/2}),$$
$$= \frac{\partial}{\partial X}\left[-\frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial X}\right)^2\left(i\gamma + \frac{a_\epsilon(t)}{2}\right) + \frac{1}{\psi}\frac{\partial \psi}{\partial t} + \frac{a_\epsilon(t)}{2}\frac{1}{\psi}\frac{\partial^2\psi}{\partial X^2}\right] + o(\epsilon^{1/2}).$$

As a consequence, equation (38) is equivalent to

$$\frac{\partial}{\partial X}\left[i2\gamma m\left[-\frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial X}\right)^2\left(i\gamma + \frac{a_\epsilon(t)}{2}\right) + \frac{1}{\psi}\frac{\partial \psi}{\partial t}\right] + \frac{a_\epsilon(t)}{2}\frac{1}{\psi}\frac{\partial^2\psi}{\partial X^2}\right] = \frac{\partial U}{\partial X}. \tag{44}$$

By integrating with respect to X, we obtain

$$i2\gamma m\left[-\frac{1}{\psi^2}\left(\frac{\partial \psi}{\partial X}\right)^2\left(i\gamma + \frac{a_\epsilon(t)}{2}\right) + \frac{1}{\psi}\frac{\partial \psi}{\partial t}\right] + \frac{a_\epsilon(t)}{2}\frac{1}{\psi}\frac{\partial^2\psi}{\partial X^2} = U(X) + \alpha(X) + o(\epsilon^{1/2}), \tag{45}$$

where $\alpha(X)$ is an arbitrary function. This concludes the proof. □

4.5. Nonlinear Schrödinger equations. — Many authors have suggested that the quantum mechanics based on the linear Schrödinger equation is only an approximation of some non linear theory with a nonlinear Schrödinger equation. Nonlinear wave mechanics was initiated by De Broglie [**11**] in order to have a better understanding of the relation between wave and particle (see [**12**],p.227-231).

Many generalizations of the Schrödinger equation exist. We can mention for example the *Staruszkievicz* [**42**] or *Bialynicki-Birula and Mycielsky* [**8**] modification. All these generalizations are not a consequence of a given principle of physic. For example, the Bialynicki-Birula and Mycielsky modification can be derived, using the hydrodynamical formalism proposed by Madelung [**34**], Bohm-Vigier [**10**] and others, by adding a *pressure term* to the Euler hydrodynamical equation (see [**41**],§.2). Of course, one can justify a posteriori a given modification by the fact that it solves some relevant problems like the *collaps* of the wave function or the Schrödinger cat paradox (see [**41**]).

Recently Castro, Mahecha and Rodriguez [**13**] have proposed a new nonlinear Schrödinger equation based on a generalization of Nottale reasoning [**37**]. In the one-dimensional case, they obtain

$$i\alpha\frac{\partial\psi}{\partial t} = -\frac{\alpha\mathrm{Re}(\alpha)}{2m}\frac{\partial^2\psi}{\partial x^2} + U(x)\psi - i\frac{\alpha\mathrm{Im}(\alpha)}{2m}\left(\frac{\partial\psi}{\partial x}\right)^2\frac{1}{\psi}, \tag{46}$$

where $\alpha \in \mathbb{C}$.

If $\mathrm{Im}(\alpha) = 0$ and $\alpha = \bar{h}$, then one recover the classical linear Schrödinger equation.

This generalization is done under two assumptions :

i) They introduce a complex "Planck constant" (i.e. α) by allowing that the normalization constant γ be complex;

ii) They consider a *complex diffusion coefficient*, which has nos counterpart in our case.

The problem is that there is no geometric interpretation of a complex diffusion coefficient, nor of a complex Planck constant. As a consequence, even if Nottale's reasoning is keep in order to derive the new non linear

Schrödinger equation (46), this is an ad-hoc mathematical generalization.

In the contrary, we have obtain a non linear Schrödinger equation (33) directly from the Scale relativity principle. All our constants have a clear geometrical meaning. Moreover, the nonlinear term is not an ad-hoc term but fixed by the theory.

5. Toward the Schrödinger equation

The generalized Schrödinger equation can be simplified in some case, by assuming that quantum-mechanical paths satisfy special regularity properties.

Theorem 5.1. — *Let $X(t)$ be a continuous non differentiable function belonging to $H^{1/2}$ such that*

$$a_\epsilon(t) = -i2\gamma. \tag{47}$$

Then, the quantized Euler-Lagrange equation takes the form

$$\gamma^2 \frac{\partial^2 \psi}{\partial X^2} + i\gamma \frac{\partial \psi}{\partial t} = (U(X,t) + \alpha(X)) \frac{\psi}{2m} + o(\epsilon^{1/2}). \tag{48}$$

We can always choose a solution of equation (48) such that

$$\alpha(X) = 0. \tag{49}$$

In this case, if

$$\gamma = \frac{\bar{h}}{2m}, \tag{SC}$$

we obtain the classical Schrödinger equation

$$i\bar{h}\frac{\partial \psi}{\partial t} + \frac{\bar{h}^2}{2m}\frac{\partial^2 \psi}{\partial X^2} = U\psi. \tag{50}$$

Proof. The only non trivial, but classical, part concerns the possibility to choose a phase of the wave function such that $\alpha(X) = 0$. The computations follow closely those of ([**4**],corollary 1).

Let ψ be a solution of (48). The basic idea is that we can always modify the phase of the wave function in order to obtain a solution of (48) such that

$\alpha(X) = 0$. So, let us consider the modified wave function

$$\tilde{\psi}(X,t) = \exp\left(i\frac{\mathcal{A}(X,t)}{2m\gamma} + \theta(X)\right) = \psi(X,t)\Theta(X). \tag{51}$$

We have

$$\begin{array}{rcl} \dfrac{\partial\tilde{\psi}}{\partial X} & = & \dfrac{\partial\psi}{\partial X}\Theta + \psi\Theta', \\ \dfrac{\partial^2\tilde{\psi}}{\partial X^2} & = & \dfrac{\partial^2\psi}{\partial X^2}\Theta + 2\dfrac{\partial\psi}{\partial X}\Theta' + \psi\Theta", \\ \dfrac{\partial\psi}{\partial t} & = & \dfrac{\partial\psi}{\partial t}\Theta, \end{array} \tag{52}$$

where $\Theta'(X)$ and $\Theta"(X)$ are the first and second derivative of Θ.

By replacing in (48), and assuming that $\alpha(X) = 0$, we obtain up to $o(\epsilon^{1/2})$ terms,

$$\Theta\left(i2\gamma m\frac{\partial\psi}{\partial t} + 2\gamma^2 m\frac{\partial^2}{\partial X^2} - U\psi\right) + 4\gamma^2 m\frac{\partial\psi}{\partial X}\Theta' + 2\gamma^2 m\psi\Theta" = 0. \tag{53}$$

As $\psi(X,t)$ is a solution of (48) with a given value $\alpha(X)$, we deduce that $\Theta(X)$ must satisfy the following ordinary differential equation

$$a(X,t)\Theta + b(X,t)\Theta' + c(X,t)\Theta" = 0, \tag{54}$$

where

$$a(X,t) = \alpha(X)\psi,\ b(X,t) = 4\gamma^2 m\frac{\partial\psi}{\partial X},\ c(X,t) = 2\gamma^2 m\psi. \tag{55}$$

This is a second order differential equation with non constant coefficients. By general theorems on linear differential equations, there always exists a solution. As a consequence, we can always choose $\Theta(X)$ such that $\phi(X,t)\Theta(X)$ satisfies (48) with $\alpha(X) = 0$. This concludes the proof. □

5.1. Difference equations and the Schrödinger condition. — The Schrödinger condition (SC) can be precised. We have the following lemma :

Lemma 5.1. — *The Schrödinger condition (SC) is equivalent to the difference equation*

$$\Delta_\epsilon^+ X(t) = \pm\sqrt{\bar{h}/m}, \tag{56}$$

and the following relations

$$\Delta_\epsilon^+ X(t) = \Delta_\epsilon^- X(t). \qquad (P)$$

Proof. By definition of $a_\epsilon(t)$ in (34), the Schrodinger condition is equivalent to the following system

$$\begin{array}{rcl} (\Delta_\epsilon^+ X(t))^2 - (\Delta_\epsilon^- X(t))^2 & = & 0, \\ (\Delta_\epsilon^+ X(t))^2 + (\Delta_\epsilon^- X(t))^2 & = & 2\bar{h}/m. \end{array} \qquad (57)$$

We deduce that $(\Delta_\epsilon^+ X(t))^2 = (\Delta_\epsilon^- X(t))^2$ and $(\Delta_\epsilon^+ X(t))^2 = \bar{h}/m$. Hence, we have $\Delta_\epsilon^+ X(t) = \pm\sqrt{\bar{h}/m}$, a constant independant of t. We obtain

$$\Delta_\epsilon^- X(t) = \Delta_\epsilon^+(t-\epsilon) = \pm\sqrt{2\bar{h}/m} = \Delta_\epsilon^+ X(t), \qquad (58)$$

which concludes the proof. □

As a consequence, we are lead to the study of difference equations of the form

$$\Delta_\epsilon^\sigma X(t) = a,\ \sigma = \pm, \qquad (59)$$

where $a \in \mathbb{R}$ is a *constant*. This kind of difference equations always have solutions of the form

$$X(t) = X^*(t) + P_\epsilon(t), \qquad (60)$$

where $X^*(t)$ is a particular solution, and $P_\epsilon(t)$ is an *arbitrary periodic* function of t of period ϵ.

Remark 5.1. — *Most of the following can be generalized to general difference equations of the form* $\Delta_\epsilon^\sigma X(t) = \Phi(t)$, *where* $\sigma = \pm$ *and* $\Phi(t)$ *is a given function (see* [**35**]*,Chap.8).*

Particular solutions always exists. Indeed, if one consider an arbitrary given function $X^*(t)$ defined on $0 \leq t < \epsilon$, then the difference equation define $X^*(t)$ at every point exterior to this interval. Of course, such kind of solutions are in general not *analytic*. We can define a special particular solution called *principal solution* following ([**35**],p.200).

For (59) the principal solution is given by (see [**35**],p.204)

$$X_c^*(t) = a(t - c - \frac{\epsilon}{2}), \tag{61}$$

where c is an arbitrary constant.

Remark 5.2. — *For $c = 0$, $\omega = 1$, $a = 1$, we obtain the Bernouilli's polynomial $B_1(t)$.*

From this section, we deduce the following *structure lemma* :

***Lemma 5.2* (structure)**. — *The Schrödinger condition is satisfied by continuous functions of the form*

$$X_c(t) = \pm\sqrt{\bar{h}/m}\,\left(t - c - \frac{\epsilon}{2}\right) + P_\epsilon(t), \tag{62}$$

where c is an arbitrary real constant, and $P_\epsilon(t)$ is an arbitrary periodic function belonging to $H^{1/2}$.

We call this set of functions, the *principal Schrödinger set*. It gives the structure of typical paths of quantum mechanics in the free case.

Remark 5.3. — *In* [**4**]*, the Schrödinger equation is obtained from a generalized equation under an assumption which is equivalent to a special fractional differential equation. In* [**7**]*, we have prove that in the framework of fractional calculus, this fractional equation has no solutions.*

References

[1] Abbott L, Wise M, Dimension of a quantum-mechanical path, Am. J. Phys. 49(1), Jan. 1981.

[2] Arnold V.I, Equations différentielles ordinaires; chapitres supplémentaires, Ed. Mir, Moscou, 1980.

[3] Ben Adda F, Geometric interpretation of the fractional derivative, Journal of the fractional calculus, Vol. 11, p.21-51, 1997.

[4] Ben Adda F, Cresson J, Quantum derivatives and the Schrödinger's equation, 19.p, to appear in Chaos, Solitons and Fractals, 2003.

[5] Ben Adda F, Cresson J, About non differentiable functions, Journ. Mathematical Analysis and Applications 263, pp. 721-737, 2001.

[6] Ben Adda. F, Cresson. J, Lois d'échelle des fonctions de classe $\mathcal{C}^\alpha$, 7.p, 2001.

[7] Ben Adda F, Cresson J, Fractional differential equations and the Schrödinger equation, 27.p, 2002.

[8] Bialynicky-Birula I, Mycielsky J, Ann. Phys. 100, 62, 1976.
[9] Bourbaki N, Algèbre I, Chap. 1-3, Hermann, Parsi, 1970.
[10] Bohm D, Vigier J, Phys. Rev. 96, 208, 1954.
[11] Broglie de L, *Non-linear wave mechanics*, Elsevier, Amsterdam, 1960.
[12] Broglie de L, *Nouvelles perspectives en microphysique*, Coll Champs Flammarion, 1992.
[13] Castro C, Mahecha J, Rodriguez B, Nonlinear QM as a fractal Brownian motion with complex diffusion constant, preprint, 2002.
[14] Cohen-Tannoudji G, Spiro M, *La matière-espace-temps. La logique des particules élémentaires*, Fayard ed., 1986.
[15] Connes A, *Noncommutative geometry*, Academic press, 1994.
[16] Connes A, Gravity coupled with matter and the foundation of noncommutative geometry, Comm. Math. Phys. 182 (1996), 155-176.
[17] Cresson J, Calcul moulien (avec un avant propos de Jean Ecalle), preprint, 2001.
[18] Cresson J, Scale relativity theory for one dimensional non differentiable manifolds, Chaos, Solitons and Fractals Vol. 14, no.4, pp. 553-562, 2002.
[19] Cresson J, Quelques problèmes de physique mathématiques, Mémoire d'habilitation à diriger des recherches, Univ de Franche-Comté, no. 99, December 2001.
[20] Cresson J, Scale geometry I. Fractal coordinates systems and scale relativity, preprint, 22.p, 2003.
[21] Cresson J, Analyse sur les variétés non différentiables, preprint, 2002.
[22] Drinfeld V, Quantum groups, Vol. 1, Proceedings of the international congress of mathematicians, Berkeley, Academic press, 1987,pp. 798-820.
[23] Drinfeld V, On quasitriangular quasi-Hopf algebra and a group closely connected with $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, Leningrad Math. J, 2(4), pp. 829-860, 1991.
[24] Einstein A, Annalen der physik 17, 549, 1905.
[25] Feynman R, L'electrodynamique quantique et l'espace-temps, Conférence Nobel in *La nature de la physique*, Ed. du Seuil, 1980.
[26] Feynman R, Hibbs A, *Quantum mechanics and path integrals*, MacGraw-Hill, 1965.
[27] Greene B, *The elegant universe, superstrings, hidden dimensions, and the quest for the utimate theory*, Norton, 1999.
[28] Hairer E, Warner G, *Analysis by its history*, Reading in Mathematics, Undergraduate texts in Math, Springer, 1996.
[29] Kolvankar K, Gangal A.D, Local fractional derivatives and fractal functions of several variables, in Proceddings of *Fractal and engineering*, 1997.
[30] Katok A, Hassenblatt B, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.
[31] Kolmogorov A, Fomine S, *Elements de la theorie des fonctions et de l'analyse fonctionelle*, Ed. Mir, Moscou.
[32] Jacobson A, *Lie algebras*, Interscience Tracts in Pure and Applied Math, 10, 1962.

[33] Lesniewski A, Noncommutative geometry, Notices of the AMS, Vol. 44, No. 7, 1997.

[34] Madelung E, Z. Physik 40, 322, 1926.

[35] Milne-Thomson L.M., *The calculus of finite differences*, Chelsea Publ. Comp., 1981.

[36] Nelson E, Phys. Rev. 150, 1079 (1966).

[37] Nottale L, *Fractal space-time and microphysics*, World Scientific, 1993.

[38] Nottale L, The theory of scale relativity, Int. J. Modern Phys. A, Vol. 7, no. 20 (1992), pp. 4899-4936.

[39] Oldham K.B, Spanier J, *The fractional calculus*, Academic press, 1974.

[40] Osler T.J, Fractional derivatives and Leibniz rule, Amer. Math. Monthly 78, 645-649, 1971.

[41] Pardy M, To the nonlinear quantum mechanics, preprint 2002, arxiv:quant-ph/0111105

[42] Puszkarz W, On the Staruszkiewicz modification of the Schrödinger equation, preprint 1999, arxiv:quant-ph/9912006

[43] Ram A, *Quantum groups : A survey of definitions, motivations, and results*, Princeton, 1997.

[44] Schrödinger E, *Physique quantique et représentation du monde*, Coll. Points Sciences, Editions du Seuil, 1992.

[45] Titschmarsh, *The theory of functions*, Oxforf University Press, 1939.

[46] Tricot C, *Courbes et dimension fractale*, 2d Ed., Springer, 1999.

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