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About fractional Hamiltonian systems

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Abstract
We follow Stanislavsky's approach of Hamiltonian formalism for fractional systems, as a model problem for the study of chaotic Hamiltonian systems. We prove that his formalism can be retrieved from the fractional embedding theory. We deduce that the fractional Hamiltonian systems of Stanislavsky stem from a particular least action principle, said to be *causal*. In this case, the fractional embedding becomes coherent.

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1. Introduction

The field of fractional calculus has been widely developing for a decade and its effectiveness has already been proved in various areas such as continuum mechanics (see [1]), chemistry (see [2]), transport theory (see [3]), fractional diffusion (see [4]), etc.

In a book by Zaslavsky [5], a link with chaotic Hamiltonian systems is drawn. Because of the appearance of fractal structures in phase spaces of nonhyperbolic Hamiltonian systems, fractional dynamics may arise in such systems. Zaslavsky then explains [5, chapters 12–13] that time takes on a fractal structure, meaning that it can be considered as a succession of specific temporal intervals. However, further investigations have to be carried out to understand and clarify the link between this peculiar temporal comportment and the fractional dynamics.

A contribution towards this is made in [6]. As a model problem for the effects of a given distribution of recurrence times on the underlying Hamiltonian dynamics, we use Stanislavsky's approach for his definition of a Hamiltonian formalism for fractional systems. Indeed, this author looks for the effects induced by the assumption that the time variable is governed by a particular stochastic process on a given Hamiltonian dynamics. This kind of process contains notably the case of the algebraic decay of recurrence times that occurs in the study of chaotic Hamiltonian systems (see [5]). Stanislavsky proves, under strong assumptions, that the induced dynamics is fractional and that the structure of the new system looks like the classical Hamiltonian one. This allows him to give a definition of a Hamiltonian formalism for fractional systems.

However, an important property of Hamiltonian systems is that they can be obtained by a variational principle, called the *Hamilton least action principle* (see [7]). A natural question with respect to Stanislavsky's construction is whether his definition of the fractional Hamiltonian system can be derived from a variational principle.

In the present paper, by using the *fractional embedding* theory developed in [8], we prove that Stanislavsky's Hamiltonian formalism for fractional systems coincides with the fractional Hamiltonian formalism induced by the fractional embedding. In particular, this means that Stanislavsky's fractional Hamiltonian systems can be obtained by a variational principle. Moreover, this fractional formalism is *coherent*, meaning that there exists a commutative diagram for the obtention of the fractional equations.

In section 2, we discuss Stanislavsky's formalism. Section 3 is devoted to the development of the fraction embedding theory using the Caputo derivatives. We obtain a causal and coherent embedding by restricting the set of variations underlying the fractional calculus of variations. We also prove that the fractional embedding of the usual Hamiltonian formalism resulting from the Lagrangian one is coherent. In section 4, we prove that the fractional Hamiltonian formalism stemming from the causal fractional embedding coincides with Stanislavsky's formalism. We finally discuss some open problems in section 5.

2. Stanislavsky's Hamiltonian formalism for fractional systems

2.1. Definition of the internal time

Let $T_1, T_2, \ldots$ be non-negative independent and identically distributed variables, with distribution ρ. We set $T(0) = 0$ and for $n \geqslant 1$, $T(n) = \sum_{i=1}^{n} T_i$. The T_i represent random temporal intervals. Let $\{N_t\}_{t\geqslant 0} = \max\{n \geqslant 0 \mid T(n) \leqslant t\}$ be the associated counting process. We suppose that there exists $0 < \alpha < 1$ such that

$$\rho(t) \sim \frac{a}{t^{1+\alpha}}, \quad t \to \infty, \quad a > 0, \quad 0 < \alpha < 1. \tag{2.1}$$

Therefore the variables T_i belong to the strict domain of attraction of an α-stable distribution. Theorem 3.2 of [9] implies

Theorem 1. *There exists a process $\{S(t)\}_{t\geqslant 0}$ and a regularly varying function b with index α such that*

$$\{b(c)^{-1} N_{ct}\}_{t\geqslant 0} \overset{FD}{\Longrightarrow} \{S(t)\}_{t\geqslant 0}, \quad c \to \infty,$$

where $\overset{FD}{\Longrightarrow}$ denotes convergence in distribution of all finite-dimensional marginal distributions.

The process $\{S(t)\}_{t\geqslant 0}$ is a hitting-time process (see [9]) and is also called a first-passage time. From [10], the distribution of $\{S(t)\}_{t\geqslant 0}$, denoted by p_t, verifies

$$\mathcal{L}[p_t](v) = \mathbb{E}[\exp(-vS(t))] = E_\alpha(-vt^\alpha),$$

where $\mathcal{L}$ is the Laplace transform and E_α is the one-parameter Mittag–Leffler function. It follows that

$$\int_0^\infty \mathrm{e}^{-wt} p_t(x)\,\mathrm{d}t = w^{\alpha-1}\exp(-xw^\alpha). \tag{2.2}$$

The process $\{S(t)\}_{t\geqslant 0}$ is increasing and may play the role of a stochastic time, which is called *internal time* in [6]. The distribution $p_t(\tau)$ represents the probability to be at the internal time τ on the real time t. Using this new time, Stanislavsky studies Hamiltonian systems which evolve according to $S(t)$.

2.2. Fractional Hamiltonian equations

We consider a Hamiltonian system, with a Hamiltonian $H(x, p)$, and associated canonical equations

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}x(t) &= \partial_2 H(x(t), p(t)), \\ \frac{\mathrm{d}}{\mathrm{d}t}p(t) &= -\partial_1 H(x(t), p(t)). \end{aligned} \tag{2.3}$$

If t is replaced by $S(t)$, how is the dynamics modified? To answer this question, Stanislavsky introduces new variables x_α and p_α defined by

$$\begin{aligned} x_\alpha(t) &= \mathbb{E}[x(S(t))] = \textstyle\int_0^\infty p_t(\tau)x(\tau)\mathrm{d}\tau, \\ p_\alpha(t) &= \mathbb{E}[p(S(t))] = \textstyle\int_0^\infty p_t(\tau)p(\tau)\mathrm{d}\tau. \end{aligned} \tag{2.4}$$

Furthermore, he assumes that

$$\begin{aligned} \partial_1 H(x_\alpha(t), p_\alpha(t)) &= \textstyle\int_0^\infty p_t(\tau)\partial_1 H(x(\tau), p(\tau))\mathrm{d}\tau, \\ \partial_2 H(x_\alpha(t), p_\alpha(t)) &= \textstyle\int_0^\infty p_t(\tau)\partial_2 H(x(\tau), p(\tau))\mathrm{d}\tau, \end{aligned} \tag{2.5}$$

which provides the following result.

Theorem 2. *Let (x, p) be a solution of (2.3). Then condition (2.5) is verified if and only if (x_α, p_α) defined by (2.4) verifies*

$$\begin{aligned} {}_0\mathcal{D}_t^\alpha\, x_\alpha(t) &= \partial_2 H(x_\alpha(t), p_\alpha(t)), \\ {}_0\mathcal{D}_t^\alpha\, p_\alpha(t) &= -\partial_1 H(x_\alpha(t), p_\alpha(t)), \end{aligned} \tag{2.6}$$

where ${}_a\mathcal{D}_t^\alpha$ is the left Caputo derivative defined by

$${}_a\mathcal{D}_t^\alpha\, f(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-\tau)^{-\alpha} f'(\tau)\,\mathrm{d}\tau.$$

Proof. As x verifies (2.3), we have

$$\int_0^\infty p_t(\tau)\partial_2 H(x(\tau), p(\tau))\,\mathrm{d}\tau = \int_0^\infty p_t(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau)\mathrm{d}\tau.$$

The Laplace transform of this expression gives

$$\begin{aligned} &\mathcal{L}\left[\int_0^\infty p_t(\tau)\partial_2 H(x, p)\,\mathrm{d}\tau\right](s) \\ &= \int_0^\infty \mathcal{L}[p_t](s)\frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau)\mathrm{d}\tau, \\ &= s^{\alpha-1}\int_0^\infty \exp(-\tau s^\alpha)\frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau)\mathrm{d}\tau \quad \text{from (2.2)}, \\ &= s^{2\alpha-1}\mathcal{L}[x](s^\alpha) - s^{\alpha-1}x(0). \end{aligned}$$

Given that $\mathcal{L}[x_\alpha](s) = s^{\alpha-1}\mathcal{L}[x](s^\alpha)$, we have

$$\mathcal{L}\left[\int_0^\infty p_t(\tau)\partial_2 H(x(\tau), p(\tau))\,\mathrm{d}\tau\right](s) = \mathcal{L}\left[{}_0\mathcal{D}_t^\alpha x_\alpha\right](s).$$

By taking the Laplace image of this relation, we obtain

$${}_0\mathcal{D}_t^\alpha\, x_\alpha(t) = \int_0^\infty p_t(\tau)\partial_2 H(x(\tau), p(\tau))\,\mathrm{d}\tau.$$

In a similar way, we also have

$${}_0\mathcal{D}_t^\alpha\, p_\alpha(t) = -\int_0^\infty p_t(\tau)\partial_1 H(x(\tau), p(\tau))\,\mathrm{d}\tau,$$

and the equivalence follows. □

For an explanation of fractional calculus and its applications, we refer the reader to [11–13].

Hence, we will say that a fractional system of the form

$$\begin{aligned} {}_0\mathcal{D}_t^\alpha\, x(t) &= f_1(x(t), p(t)), \\ {}_0\mathcal{D}_t^\alpha\, p(t) &= f_2(x(t), p(t)) \end{aligned}$$

is Hamiltonian in the sense of Stanislavsky if there exists a function $H(x, p)$ such that

$$\begin{aligned} f_1(x, p) &= \partial_2 H(x, p), \\ f_2(x, p) &= -\partial_1 H(x, p). \end{aligned}$$

We can see that the fractional derivative ${}_0\mathcal{D}_t^\alpha$ appears as a natural consequence of the structure of the internal time $S(t)$. The fractional exponent α is exactly determined by the behaviour (2.1) of long time intervals. We note that if we had $\alpha \geqslant 1$ in (2.1), the α-stable distribution would be the Gaussian one, we would have $p_t(\tau) = \delta_\tau(t)$ and then $S(t) \equiv t$. In this case, internal time and real time would be the same. Consequently, for $\alpha \geqslant 1$, the associated derivative is the classical one.

3. Fractional embedding of Lagrangian and Hamiltonian systems

An important property of classical Hamiltonian systems is that they are solutions of a variational principle, called the *Hamilton least action principle* (see [7]). A natural question is whether the fractional Hamiltonian systems defined by Stanislavsky can be derived from a variational principle.

Fractional Euler–Lagrange and Hamilton equations have been first derived in [14], in order to include frictional forces into a variational principle. In [15], a fractional Euler–Lagrange equation is obtained using a fractional least action principle. This formalism includes the left and right fractional derivatives. The related Hamilton equations are derived in [16]. However, their equations are different from those obtained by Stanislavsky.

Using the fractional embedding theory developed in [8], we prove that the Stanislavsky Hamiltonian formalism stems from a fractional variational principle, called *causal*, and moreover that this construction is coherent.

We sum up here the general ideas of the fractional embedding theory for the Caputo derivative. Similarly to the left one, the right Caputo derivative is defined by

$$ {}_t\mathcal{D}_b^\alpha\ f(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b (\tau - t)^{-\alpha} f'(\tau)\,\mathrm{d}\tau. $$

The left fractional integral is defined by

$$ {}_a\mathcal{D}_t^{-\alpha}\ f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau $$

and the right one by

$$ {}_t\mathcal{D}_b^{-\alpha}\ f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} f(\tau)\,\mathrm{d}\tau. $$

3.1. Fractional embedding of differential operators

Let $\mathbf{f} = (f_1, \ldots, f_p)$ and $\mathbf{g} = (g_1, \ldots, g_p)$ be two p-uplets of smooth functions $\mathbb{R}^{k+2} \longrightarrow \mathbb{R}^l$. Let $a, b \in \mathbb{R}$ with $a < b$. We denote by $\mathcal{O}(\mathbf{f}, \mathbf{g})$ the differential operator defined by

$$ \mathcal{O}(\mathbf{f},\mathbf{g})(x)(t) = \sum_{i=0}^{p} \left(f_i \cdot \frac{\mathrm{d}^i}{\mathrm{d}t^i} g_i \right) \left(x(t), \ldots, \frac{\mathrm{d}^k}{\mathrm{d}t^k} x(t), t \right), \tag{3.1} $$

where, for any functions $y = (y^1, \ldots, y^l)$, $z = (z^1, \ldots, z^l) : \mathbb{R} \longrightarrow \mathbb{R}^l$,

$$ (y \cdot z)(t) = (y^1(t)\, z^1(t), \ldots, y^l(t)\, z^l(t)). $$

The fractional embedding of $\mathcal{O}(\mathbf{f}, \mathbf{g})$, denoted by $\mathcal{E}_\alpha(\mathcal{O}(\mathbf{f}, \mathbf{g}))$, is defined by

$$ \mathcal{E}_\alpha(\mathcal{O}(\mathbf{f},\mathbf{g}))(x)(t) = \sum_{i=0}^{p} (f_i \cdot ({}_a\mathcal{D}_t^\alpha)^i g_i)(x(t), \ldots, ({}_a\mathcal{D}_t^\alpha)^k x(t), t). \tag{3.2} $$

We define the ordinary differential equation associated with $\mathcal{O}(\mathbf{f}, \mathbf{g})$ by

$$ \mathcal{O}(\mathbf{f},\mathbf{g})(x) = 0. \tag{3.3} $$

The fractional embedding $\mathcal{E}_\alpha(\mathcal{O}(\mathbf{f}, \mathbf{g}))$ of (3.3) is defined by

$$ \mathcal{E}_\alpha(\mathcal{O}(\mathbf{f},\mathbf{g}))(x) = 0. $$

3.2. Lagrangian systems

Now we consider a Lagrangian system, with a smooth Lagrangian $L(x, v, u)$ and $u \in [a, b]$. The Lagrangian L can naturally lead to a differential operator of the form (3.1):

$$ \mathcal{O}(1, L)(x)(t) = L\left(x(t), \frac{\mathrm{d}}{\mathrm{d}t} x(t), t \right). $$

Now we identify L and $\mathcal{O}(1, L)$. The fractional embedding (3.2) of L, $\mathcal{E}_\alpha(L)$, is hence given by

$$ \mathcal{E}_\alpha(L)(x)(t) = L(x(t), {}_a\mathcal{D}_t^\alpha x(t), t). $$

In Lagrangian mechanics, the action and its extrema play a central role. For any mapping g, the action of g, denoted by $\mathcal{A}(g)$, is defined by

$$ \mathcal{A}(g)(x) = \int_a^b g(x)(t)\,\mathrm{d}t. $$

For example, with the identification $L \equiv \mathcal{O}(1, L)$, the action of L is given by

$$ \mathcal{A}(L)(x) = \int_a^b L\left(x(t), \frac{\mathrm{d}}{\mathrm{d}t} x(t), t \right) \mathrm{d}t, $$

and concerning the fractional embedding of L, the associated action is

$$ \mathcal{A}(\mathcal{E}_\alpha(L))(x) = \int_a^b L\left(x(t), {}_a\mathcal{D}_t^\alpha x(t), t \right) \mathrm{d}t. $$

The extrema of the action of a Lagrangian L provide the equation of motion associated.

Theorem 3. *If the action $\mathcal{A}(L)$ is extremal in x, then x satisfies the Euler–Lagrange equation, given by*

$$ \partial_1 L\left(x(t), \frac{\mathrm{d}}{\mathrm{d}t} x(t), t \right) - \frac{\mathrm{d}}{\mathrm{d}t} \partial_2 L\left(x(t), \frac{\mathrm{d}}{\mathrm{d}t} x(t), t \right) = 0. \tag{3.4} $$

This equation is denoted by $EL(L)$.

This procedure should not be modified with fractional derivatives. Indeed, the strict definition of the Lagrangian L does not involve any temporal derivative. The dynamics is afterwards fixed with the choice of the derivative $\mathcal{D}$ and the relation $v(t) = \mathcal{D}x(t)$. The variational principle providing the Euler–Lagrange equation uses an integration by parts, which remains in the fractional case:

$$\begin{aligned}\int_a^b \left[{}_a\mathcal{D}_t^\alpha f(t)\right] g(t)\mathrm{d}t = \int_a^b f(t)\left[{}_b\mathcal{D}_t^\alpha g(t)\right]\mathrm{d}t \\ + g(b)\,{}_a\mathcal{D}_b^{-(1-\alpha)} f(b) \\ - f(a)\,{}_a\mathcal{D}_b^{-(1-\alpha)} g(a).\end{aligned}$$

We introduce the space of variations

$$V_\alpha = \left\{h \in C^1([a,b]) \mid {}_a\mathcal{D}_b^{-(1-\alpha)} h(a) = h(b) = 0\right\}.$$

For $h \in V_\alpha$, we have

$$\begin{aligned}\mathcal{A}(\mathcal{E}_\alpha(L))(x+h) = \mathcal{A}(\mathcal{E}_\alpha(L))(x) \\ + \int_a^b [\partial_1 L + {}_t\mathcal{D}_b^\alpha\, \partial_2 L](x(t), {}_a\mathcal{D}_t^\alpha\, x(t), t) \\ \times h(t)\,\mathrm{d}t + o(h),\end{aligned}$$

which implies that the differential of $\mathcal{A}(\mathcal{E}_\alpha(L))$ in x is given, for any $h \in V_\alpha$, by

$$\begin{aligned}\mathrm{d}\mathcal{A}(\mathcal{E}_\alpha(L))(x,h) &= \int_a^b [\partial_1 L + {}_t\mathcal{D}_b^\alpha\, \partial_2 L](x, {}_a\mathcal{D}_t^\alpha\, x, t)\, h(t)\,\mathrm{d}t, \\ &= \langle [\partial_1 L + {}_t\mathcal{D}_b^\alpha\, \partial_2 L](x(\cdot), {}_a\mathcal{D}_t^\alpha\, x(\cdot), \cdot), h\rangle,\end{aligned}$$

where $\langle f, g\rangle = \int_a^b f(t)g(t)\,\mathrm{d}t$ is a scalar product defined on $C^1([a,b])$.

If $E \subset V_\alpha$, we will say that $\mathcal{A}(\mathcal{E}_\alpha(L))$ is E-extremal in x if for all $h \in E$, $\mathrm{d}\mathcal{A}(\mathcal{E}_\alpha(L))(x,h) = 0$.

So we obtain a first fractional Euler–Lagrange equation with the following result.

Theorem 4. *$\mathcal{A}(\mathcal{E}_\alpha(L))$ is V_α-extremal in x if and only if x verifies*

$$\partial_1 L(x(t), {}_a\mathcal{D}_t^\alpha\, x(t), t) + {}_t\mathcal{D}_b^\alpha\, \partial_2 L(x(t), {}_a\mathcal{D}_t^\alpha\, x(t), t) = 0. \tag{3.5}$$

Proof. $\mathcal{A}(\mathcal{E}_\alpha(L))$ is V_α-extremal in x if and only if for all $h \in V_\alpha$,

$$\langle [\partial_1 L + {}_t\mathcal{D}_b^\alpha\, \partial_2 L](x(\cdot), {}_a\mathcal{D}_t^\alpha\, x(\cdot), \cdot), h\rangle = 0.$$

This is equivalent to $[\partial_1 L + {}_t\mathcal{D}_b^\alpha\, \partial_2 L](x(\cdot), {}_a\mathcal{D}_t^\alpha\, x(\cdot)) \in V_\alpha^\perp$. We conclude by noting that $V_\alpha^\perp = \overline{V_\alpha}^\perp = \{0\}$, where $\overline{V_\alpha}$ is the adherence of V_α in $C^1([a,b])$, equal to $C^1([a,b])$ entirely. □

Equation (3.5) will be called the *general fractional Euler–Lagrange equation* and will be denoted by $EL_g(\mathcal{E}_\alpha(L))$. Contrary to (3.4), two operators are involved here. We will now discuss the problematic presence of ${}_t\mathcal{D}_b^\alpha$.

3.3. Coherence and causality

Because of the simultaneous presence of the two derivatives, the position of x at time t depends on its past positions, through ${}_a\mathcal{D}_t^\alpha$, but also on its future ones, through ${}_t\mathcal{D}_b^\alpha$. The principle of causality is here violated, which seems crippling from a physical point of view. Moreover, we note that (3.4) can be written in the form (3.3), with $\mathbf{f} = (1, 1)$ and $\mathbf{g} = (\partial_1 L, -\partial_2 L)$. The fractional embedding $\mathcal{E}_\alpha(EL(L))$ of (3.4) is therefore

$$\partial_1 L(x(t), {}_a\mathcal{D}_t^\alpha\, x(t), t) - {}_a\mathcal{D}_t^\alpha\, \partial_2 L(x(t), {}_a\mathcal{D}_t^\alpha\, x(t), t) = 0,$$

which shows that $EL_g(\mathcal{E}_\alpha(L)) \not\equiv \mathcal{E}_\alpha(EL(L))$: fractional embedding and the least action principle are not commutative. So we obtain two procedures providing different fractional equations, which also seems unsatisfactory. We are facing a Cornelian choice: shall we preserve causality or the least action principle? A possible way of solving this problem is to restrict the space of variations.

We note $\tilde{V}_\alpha = \{h \in V_\alpha \mid {}_a\mathcal{D}_t^\alpha\, h = -{}_t\mathcal{D}_b^\alpha\, h\}$ and $K_\alpha = {}_a\mathcal{D}_t^\alpha + {}_t\mathcal{D}_b^\alpha$, defined on $C^1([a,b])$. For any $f, g \in V_\alpha$, $\langle K_\alpha f, g\rangle = \langle f, K_\alpha g\rangle$. We show that K_α is essentially self-adjoint and we obtain a new Euler–Lagrange equation:

Theorem 5. *$\mathcal{A}(\mathcal{E}_\alpha(L))$ is $\tilde{V}_\alpha$-extremal in x if and only if there exists a function g such that x verifies*

$$\partial_1 L(x(t), {}_a\mathcal{D}_t^\alpha\, x(t), t) - {}_a\mathcal{D}_t^\alpha\, \partial_2 L(x(t), {}_a\mathcal{D}_t^\alpha\, x(t), t) = K_\alpha\, g.$$

Proof. $\mathcal{A}(\mathcal{E}_\alpha(L))$ is $\tilde{V}_\alpha$-extremal in x if and only if $[\partial_1 L + {}_t\mathcal{D}_b^\alpha\, \partial_2 L](x(\cdot), {}_a\mathcal{D}_t^\alpha\, x(\cdot)) \in \tilde{V}_\alpha^\perp$. Given that $\tilde{V}_\alpha^\perp = (\mathcal{K}er\ K_\alpha)^\perp = \mathcal{I}m\ K_\alpha$, $\mathcal{A}(\mathcal{E}_\alpha(L))$ is extremal if and only if there exists $\tilde{g}$ such that

$$\partial_1 L(x(t), {}_a\mathcal{D}_t^\alpha\, x(t), t) + {}_t\mathcal{D}_b^\alpha\, \partial_2 L(x(t), {}_a\mathcal{D}_t^\alpha\, x(t), t) = K_\alpha\, \tilde{g}.$$

We conclude by setting $g(t) = \tilde{g}(t) + \partial_2 L(x(t), {}_a\mathcal{D}_t^\alpha\, x(t), t)$. □

Restricting the space of variations breaks the unicity of the solution. However, among those solutions, there is a single one that remains causal (without the operator ${}_t\mathcal{D}_b^\alpha$), for $g = 0$:

$$\partial_1 L(x(t), {}_a\mathcal{D}_t^\alpha\, x(t), t) - {}_a\mathcal{D}_t^\alpha\, \partial_2 L(x(t), {}_a\mathcal{D}_t^\alpha\, x(t), t) = 0. \tag{3.6}$$

Equation (3.6) will be called the *causal fractional Euler–Lagrange equation*, and will be denoted by $EL_c(\mathcal{E}_\alpha(L))$.

Now causality is respected and we have $EL_c(\mathcal{E}_\alpha(L)) \equiv \mathcal{E}_\alpha(EL(L))$. In this case, the fractional embedding is called *coherent*, in the sense that the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\mathcal{E}_\alpha"] \arrow[d, "\text{(causal) LAP}"'] & \mathcal{E}_\alpha(L) \arrow[d, "\text{causal FLAP}"] \\ (\partial_1 L - \frac{\mathrm{d}}{\mathrm{d}t}\partial_2 L) = 0 \arrow[r, "\mathcal{E}_\alpha"'] & (\partial_1 L - {}_a\mathcal{D}_t^\alpha\, \partial_2 L) = 0\,. \end{tikzcd}$$

The abbreviation (F)LAP stands for the '(fractional) least action principle'. As ${}_a\mathcal{D}_t^1 = -{}_t\mathcal{D}_b^1 = \mathrm{d}/\mathrm{d}t$, we can say that the least action is also causal in the classical case.

However, in the fractional case, the physical meaning of $\tilde{V}_\alpha$ is not clear, but it might be related to a reversible dynamics of the variations. Furthermore, this underlines the significant role of variations in the global dynamics.

3.4. Fractional Hamiltonian systems based on fractional Lagrangian ones

There exists a natural derivation of a Hamiltonian system from a Lagrangian system based on the Legendre transformation. We consider an autonomous Lagrangian system, with Lagrangian $L(x, v)$, and we suppose that

$$\forall x,\ v \mapsto \partial_2 L(x, v) \quad \text{is bijective.}$$

The momentum associated with the variable x is $p = \partial_2 L(x, v)$. So there exists a mapping f named Legendre transformation such that $v = f(x, p)$. The Hamiltonian H associated with L is defined by

$$H(x, p) = pf(x, p) - L(x, f(x, p)).$$

It implies $\partial_1 H(x, p) = -\partial_1 L(x, f(x, p))$ and $\partial_2 H(x, p) = f(x, p)$.

We introduce the function

$$F_{\mathrm{LH}}(x, p, v, w) = \begin{pmatrix} p - \partial_2 L(x, v) \\ \partial_1 H(x, p) + \partial_1 L(x, f(x, p)) \\ \partial_2 H(x, p) - f(x, p) \end{pmatrix}.$$

The link between the Lagrangian and Hamiltonian formalisms is done through the equation

$$F_{\mathrm{LH}}(x, p, v, w) = 0. \tag{3.7}$$

The momentum p induces a function $p(t) = \partial_2 L(x(t), v(t))$, which can be considered as the dynamical momentum.

For the classical dynamics, (3.7) becomes

$$F_{\mathrm{LH}}\left(x(t), p(t), \frac{\mathrm{d}}{\mathrm{d}t}x(t), \frac{\mathrm{d}}{\mathrm{d}t}p(t)\right) = 0,$$

i.e.

$$p(t) = \partial_2 L\left(x(t), \frac{\mathrm{d}}{\mathrm{d}t}x(t)\right),$$

$$\partial_1 H(x(t), p(t)) = -\,\partial_1 L\left(x(t), \frac{\mathrm{d}}{\mathrm{d}t}x(t)\right),$$

$$\partial_2 H(x(t), p(t)) = \frac{\mathrm{d}}{\mathrm{d}t}x(t).$$

Moreover, if $x(t)$ is the solution of the Euler–Lagrange equation (3.4), we obtain the canonical equations

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \partial_2 H(x(t), p(t)),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = -\,\partial_1 H(x(t), p(t)).$$

For the fractional case, the fractional embedding (3.2) of (3.7) is given by

$$p(t) = \partial_2 L\left(x(t), {}_a\mathcal{D}_t^\alpha\, x(t)\right),$$
$$\partial_1 H(x(t), p(t)) = -\,\partial_1 L\left(x(t), {}_a\mathcal{D}_t^\alpha\, x(t)\right),$$
$$\partial_2 H(x(t), p(t)) = {}_a\mathcal{D}_t^\alpha\, x(t).$$

Then the following result is obtained:

Theorem 6. *If $x(t)$ is the solution of the causal fractional Euler–Lagrange equation (3.6), then we have*

$${}_a\mathcal{D}_t^\alpha\, x(t) = \partial_2 H(x(t), p(t)),$$
$${}_a\mathcal{D}_t^\alpha\, p(t) = -\,\partial_1 H(x(t), p(t)).$$

These are the equations describing the dynamics of a fractional Hamiltonian system derived from a Lagrangian formalism. But Hamiltonian systems can also be considered directly, as will be seen.

3.5. Embedded Hamiltonian systems

Now we consider a Hamiltonian system as defined in section 2, with a Hamiltonian $H(x, p)$ and with equations (2.3) associated. The fractional embedding (3.2) of (2.3) is

$$\begin{aligned} {}_a\mathcal{D}_t^\alpha\, x(t) &= \partial_2 H(x(t), p(t)), \\ {}_a\mathcal{D}_t^\alpha\, p(t) &= -\partial_1 H(x(t), p(t)). \end{aligned} \tag{3.8}$$

Furthermore, by introducing the function

$$L_{\mathrm{H}}(x, p, v, w) = pv - H(x, p),$$

we can verify that the classical Hamiltonian equations are given by the extrema of the action of L_{H} defined by

$$\mathcal{A}(L_{\mathrm{H}})(x, p) = \int_a^b L_{\mathrm{H}}\left(x(t), p(t), \frac{\mathrm{d}}{\mathrm{d}t}x(t), \frac{\mathrm{d}}{\mathrm{d}t}p(t)\right) \mathrm{d}t.$$

In the fractional case, the action becomes

$$\mathcal{A}(\mathcal{E}_\alpha(L_{\mathrm{H}}))(x, p) = \int_a^b L_{\mathrm{H}}\left(x(t), p(t), {}_a\mathcal{D}_t^\alpha\, x(t), {}_a\mathcal{D}_t^\alpha\, p(t)\right) \mathrm{d}t.$$

Using the causal fractional Euler–Lagrange equation for L_{H}, we obtain the following result.

Theorem 7 (Hamiltonian coherence). *Let H be a Hamiltonian function. The solutions $(x(t), p(t))$ of the fractional system (3.8) coincide with causal extremal points of the action $\mathcal{A}(\mathcal{E}_\alpha(L_H))$. More precisely, the following diagram commutes:*

$$\begin{tikzcd} L_H \arrow[r, "\mathcal{E}_\alpha"] \arrow[d, "\text{(causal) LAP}"'] & \mathcal{E}_\alpha(L_H) \arrow[d, "\text{causal FLAP}"] \\ \frac{\mathrm{d}}{\mathrm{d}t}(x(t),p(t)) = (\partial_2 H, -\partial_1 H) \arrow[r, "\mathcal{E}_\alpha"'] & {}_a\mathcal{D}_t^\alpha(x(t),p(t)) = (\partial_2 H, -\partial_1 H) \end{tikzcd}$$

Proof. The causal fractional Euler–Lagrange equation for L_{H} is

$$-\partial_1 H(x(t), p(t)) - {}_a\mathcal{D}_t^\alpha\, p(t) = 0,$$
$${}_a\mathcal{D}_t^\alpha\, x(t) - \partial_2 H(x(t), p(t)) = 0,$$

which is exactly (3.8). □

So we have coherence between the directly embedded equations and the equations obtained by a variational principle. But we also have coherence between this section and the previous one, i.e. between the fractional Hamiltonian systems resulting from the Lagrangian ones and the embedded Hamiltonian systems.

In other words, the equivalent approaches for Hamiltonian systems in the classical case remain equivalent in the fractional case if we use causal variational principles.

Now we will discuss the link between this formalism and Stanislavsky's formalism.

4. Compatibility between the two formalisms

Condition (2.5) means that the partial derivatives of H commute with $\mathbb{E}[\cdot(S(t))]$. This condition just seems to be of technical order and appears to be unrelated to the real dynamics. However, by using the fractional embedding, we can identify the underlying dynamical link in the case of natural Lagrangian systems.

We consider a natural Lagrangian system, i.e. with a Lagrangian L of the form $L(x, v) = \frac{1}{2}mv^2 - U(x)$, and the Hamiltonian $H(x, p)$ derived as in section 3.4. We choose $a = 0$ for the initial instant. We have $H(x, p) = \frac{1}{2m}p^2 + U(x)$, with $p = \partial_2 L(x, v) = mv$. We suppose that (x, p) is the solution of the classical Hamiltonian equations (2.3). We define the associated variables x_α and p_α by (2.4).

Theorem 8. *If x_α is the solution of the causal fractional Euler–Lagrange equation (3.6) associated with L, then condition (2.5) is verified. Consequently, (x_α, p_α) is the solution of (2.6).*

Proof. We set $\tilde{p}_\alpha(t) = \partial_2 L(x_\alpha(t), {}_0\mathcal{D}_t^\alpha\, x_\alpha(t))$, i.e. $\tilde{p}_\alpha(t) = m\, {}_0\mathcal{D}_t^\alpha\, x_\alpha(t)$. Then, from theorem 6, $(x_\alpha, \tilde{p}_\alpha)$ is the solution of

$$
\begin{aligned}
{}_0\mathcal{D}_t^\alpha\, x_\alpha(t) &= \partial_2 H(x_\alpha(t), \tilde{p}_\alpha(t)), \\
{}_0\mathcal{D}_t^\alpha\, \tilde{p}_\alpha(t) &= -\,\partial_1 H(x_\alpha(t), \tilde{p}_\alpha(t)).
\end{aligned}
\tag{4.1}
$$

Moreover, we have

$$
\begin{aligned}
\tilde{p}_\alpha(t) &= m\, {}_0\mathcal{D}_t^\alpha \int_0^\infty p_t(\tau) x(\tau)\, \mathrm{d}\tau = m \int_0^\infty p_t(\tau) \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)\, \mathrm{d}\tau \\
&= \int_0^\infty p_t(\tau) m v(\tau)\, \mathrm{d}\tau = \int_0^\infty p_t(\tau) p(\tau)\, \mathrm{d}\tau = p_\alpha(t).
\end{aligned}
$$

So we can replace $\tilde{p}_\alpha$ by p_α in (4.1), to obtain (2.6). We conclude by using theorem 2. □

5. Conclusion

If we consider the temporal evolution variable of a Lagrangian system as a succession of random intervals and if their density has a power-law tail, then the dynamics of this system is fractional. The associated equations can be determined through a fractional embedding, based on a least action principle. In order to obtain causal and coherent equations, it is necessary to restrict the space of variations. This condition might be seen as a way to cancel the finalist aspect of the least action principle. Even if it is still unclear, this model of time could notably be appropriated for the description of some chaotic Hamiltonian dynamics. Some numerical experiments show that distributions of Poincaré recurrence times may possess a power-law tail (see [5, chapter 11], [17, 18]). Consequently, the time may be decomposed into a succession of recurrence times. For long time scale dynamics, the number of intervals is great and the new characteristic time clock may become $S(t)$. This new time takes into account the peculiar structure of the recurrence times: if the power-law exponent α verifies $0 < \alpha < 1$, the long time scale dynamics becomes fractional with the same exponent α. This idea of stacked dynamics based on two time scales could be linked with [19], where close results are obtained. However, because of the Kac lemma ([20]), which states that the mean recurrence time is finite, condition (2.1) may be valid only locally, near some island boundaries, called sticky zones. Further investigations have to be carried out to clarify this point.

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