VARIATIONAL FORMULATIONS OF DIFFERENTIAL EQUATIONS AND ASYMMETRIC FRACTIONAL EMBEDDING

by

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Abstract. — Variational formulations for classical dissipative equations, namely friction and diffusion equations, are given by means of fractional derivatives. In this way, the solutions of those equations are exactly the extremal of some fractional Lagrangian actions. The formalism used is a generalization of the fractional embedding developed by Cresson ["Fractional embedding of differential operators and Lagrangian systems", J. Math. Phys. 48, 033504 (2007)], where the functional space has been split in two in order to take into account the asymmetry between left and right fractional derivatives. Moreover, this asymmetric fractional embedding is compatible with the least action principle and respects the physical causality principle.

Keywords : Least action principle, Calculus of variations, Fractional calculus, Classical mechanics, Dynamical systems, Differential equations.

Contents

1. Introduction

Fractional differential equations may be used to describe various phenomena, for instance in viscoelasticity [**4**], anomalous diffusion [**25**] notably in chaotic systems [**35**], or in phase transitions [**17**]. Overviews on applications of fractional calculus may be found in the books [**18, 31**]. In this paper, we are interested in two different problems where fractional differential equations arise: direct fractional generalization of classical differential equations and F. Riewe's [**29, 30**] approach to classical dissipative systems via fractional Lagrangian systems.

1.1. Direct fractional generalization of differential equations. — Most of fractional equations are obtained from a classical equation, like the wave equation or the diffusion equation, by replacing the time derivative by a fractional derivative. As an example, we refer to [**33**] and [**34**] for the introduction of the fractional wave equation and the fractional diffusion equation respectively. However, the status of such ad-hoc generalizations is not so easy to interpret. The main problem being that these generalizations are not stable under change of variables. Precisely, if we denote by (E) a given differential equation in a coordinates system x, we obtain under a change of variable $y = h(x)$, where h is a C^1-diffeomorphism, a new equation denoted by (E') whose flow ψ_t is related to the flow ϕ_t of (E) by the conjugacy relation $\psi = h \circ \phi \circ h^{-1}$. However, if we perform a fractional generalization of (E) and (E') denoted by $(E^\alpha\ )$ and (E'^α) respectively and we denote by ψ_t^α and ψ_t^α the corresponding flow, there exists in general no conjugacy relation between ψ_t^α and ϕ_t^α. As a consequence, the meaning of this extension is not clear.

It is then important to derive fractional generalization of differential equations on a more intrinsic (i.e. coordinates independent) way. The problem can be first studied for some equations possessing a specific structures like Lagrangian or Hamiltonian systems, or symmetries properties. These properties are indeed intrinsic. An idea is then to base a generalization on these structures dealing with fractional generalization of Lagrangian or Hamiltonian systems.

1.2. F. Riewe's approach to dissipative dynamical systems. — For dissipative systems a classical result of P.S. Bauer [**7**] in 1931 stated that a linear set of differential equations with constant coefficients cannot be derived from a variational principle. The main obstruction is precisely the dissipation of energy which induces a dynamic which is not reversible in time. H. Bateman [**6**] pointed out that this obstruction is only valid if one understand that the variational principle does not produce additional equations. In particular, H. Bateman construct a complementary set of equations which enables him to find a variational formulation. The main idea behind Bateman's approach is that a dissipative system must be seen as physically incomplete. An extension of Bateman's construction has been recently given for nonlinear evolution equations [**16**].

In this paper, we follow a different approach initiated by F. Riewe ([**29**],[**30**]) in 1996-1997. He defined a *fractional Lagrangian* framework to deal with dissipative systems. Riewe's theory follows from a simple observation : "*If the Lagrangian contains terms proportional to* $\left(\frac{d^n x}{dt^n}\right)^2$*, then the Euler-Lagrange equation will have a term proportional to* $\frac{d^{2n}x}{dt^{2n}}$*. Hence a frictional force* $\gamma\frac{dx}{dt}$ *should follow from a Lagrangian containing a term proportional to the fractional derivative* $\left(\frac{dx^{1/2}}{dt^{1/2}}\right)^2$." where the notation $\frac{d^{1/2}}{dt^{1/2}}$ represents formally an operator satisfying the composition rule $\frac{d^{1/2}}{dt^{1/2}} \circ \frac{d^{1/2}}{dt^{1/2}} = \frac{d}{dt}$. He then studied fractional Lagrangian functionals using the left and right Riemann-Liouville fractional derivatives which satisfy the composition rule formula. A simple example of functional studied by Riewe is given by

$$\mathcal{L}(x) = \int_a^b L(x, \mathcal{D}_+^{1/2} x, \mathcal{D}_-^{1/2} x, dx/dt) dt,$$

where $x : [a, b] \to \mathbb{R}$ and $L : \begin{cases} \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} & \to & \mathbb{R}, \\ (x, v_+, v_-, v) & \longmapsto & L(x, v_+, v_-, v). \end{cases}$ He proved that critical points x of this functional corresponds to the solutions of the generalized fractional Euler-Lagrange equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v}(\star^{1/2}) \right) + \mathcal{D}_-^{1/2} \left(\frac{\partial L}{\partial v_+}(\star^{1/2}) \right) * \mathcal{D}_+^{1/2} \left(\frac{\partial L}{\partial v_-}(\star^{1/2}) \right) = \frac{\partial L}{\partial x}(\star^{1/2}),$$

where $\star^{1/2} = (x, \mathcal{D}_+^{1/2} x, \mathcal{D}_-^{1/2} x, dx/dt)$. Riewe derived such a generalized Euler-Lagrange equation for more general functionals depending on left and right Riemann-Liouville derivatives $\mathcal{D}_-^\alpha$ and $\mathcal{D}_+^\alpha$ with arbitrary $\alpha > 0$ (see [**29**], equation (45) p. 1894). The main property of this Euler-Lagrange equation is that the dependence of L with respect to $\mathcal{D}_+^{1/2}$ (resp. $\mathcal{D}_-^{1/2}$) induces a derivation with respect to $\mathcal{D}_-^{1/2}$ (resp. $\mathcal{D}_+^{1/2}$) in the equation. As a consequence, we will always obtain mixed terms of the form $\mathcal{D}_-^{1/2} \circ \mathcal{D}_+^{1/2} x$ or $\mathcal{D}_+^{1/2} \circ \mathcal{D}_-^{1/2} x$. For example, if we consider the Lagrangian

$$L(x, v_+, v_-, v) = \frac{1}{2} m v^2 - U(x) + \frac{1}{2} \gamma v_+^2,$$

we obtain as a generalized Euler-Lagrange equation

$$m \frac{d^2 x}{dt^2} + \gamma \mathcal{D}_-^{1/2} \circ \mathcal{D}_+^{1/2} x + U'(x) = 0.$$

However, in general

$$\mathcal{D}_-^{1/2} \circ \mathcal{D}_+^{1/2} x \neq \frac{dx}{dt},$$

so that this theory can not be used in order to provide a variational principle for the linear friction problem. This problem of the mixing between the left and right derivative in the fractional calculus of variations is well known (see for example Agrawal [**1**]). It is due to the *integration by parts* formula which is given for f and g in $C^0([a, b])$ by

$$\int_a^b f(t).\, \mathcal{D}_-^\alpha g(t) dt = \int_a^b \mathcal{D}_+^\alpha f(t).g(t) dt.$$

In ([**29**],p.1897) Riewe considered the limit $a \to b$ while keeping $a < b$. He then approximated $\mathcal{D}_-^\alpha$ by $\mathcal{D}_+^\alpha$. However, this approximation is not justified in general for a large class of function so that Riewe's derivation of a variational principle for the linear friction problem is not valid.

1.3. Abstract embedding of differential equations and Lagrangian systems. — The two previous problems can be studied in the framework of embeddings initiated in [**10**] and developed further in [**8, 9, 14, 21**]. In the following, we discuss arbitrary extension of differential equations and Lagrangian systems. We specialize the discussion to the fractional case in the next Subsection. An embedding is made of two parts :

- An algebraic part which formalizes the fact to replace a given time derivative by a generalized one denoted $\mathcal{D}$ in a given differential equation. The starting point for this generalization is the differential operator $\mathcal{O}$ which depends on the classical derivative d/dt and its iterations related to the differential equation $\mathcal{O}(x) = 0$. By using a new derivative $\mathcal{D}$, we first extend the operator $\mathcal{O}$ to a new one, $\mathcal{E}_{\mathcal{D}}(\mathcal{O})$ and obtain a new equation, $\mathcal{E}_{\mathcal{D}}(\mathcal{O})(x) = 0$. The following diagram sums up this procedure:

$$\begin{tikzcd} \mathcal{O}(x)=0 \arrow[d] & \mathcal{E}_{\mathcal{D}}(\mathcal{O})(x)=0 \\ \mathcal{O} \arrow[r, "\mathcal{E}_{\mathcal{D}}"] & \mathcal{E}_{\mathcal{D}}(\mathcal{O}) \arrow[u] \end{tikzcd} \tag{1.1}$$

This algebraic manipulation is classical in analysis when one extends partial differential equations to Schwartz's distribution using the symbol of the underlying differential operator (see for example [?]).

In the following, we are interested in Lagrangian systems defined by a Lagrangian $L(x(t), \frac{d}{dt}x(t), t)$ whose dynamics is given by a second order differential equation defined by

$$(EL) \qquad \partial_1 L\left(x(t), \frac{d}{dt}x(t), t\right) - \frac{d}{dt}\,\partial_2 L\left(x(t), \frac{d}{dt}x(t), t\right) = 0,$$

called the Euler-Lagrange equation. Equation (EL) plays the role of $\mathcal{O}(x) = 0$ in diagram (1.1), and leads to an embedded Euler-Lagrange equation, denoted by $\mathcal{E}_{\mathcal{D}}(EL)$.

- An analytic part consisting in extending Lagrangian functionals. As we previously said, it is important to base a generalization not on the differential equation itself but on a more intrinsic structure associated to this equation. In the case of Lagrangian systems, one can prove that x is a solution of the Euler-Lagrange equation if and only if its is an extremal of the functional

$$\mathcal{A}(L)\,:\, x \mapsto \int_a^b L\left(x(t), \frac{d}{dt}x(t), t\right)\, dt,$$

called the action in classical mechanics. Such a variational principle is called the least-action principle in classical mechanics. The underlying framework is the classical calculus of variations. Extremals are obtained looking for the variations of the functional over a particular set denoted by $\mathcal{V}$ in the following.

Embedding the Lagrangian L also makes sense and provides a new function $\mathcal{E}_{\mathcal{D}}(L)$. We can then define a generalized Lagrangian functional $\mathcal{A}(\mathcal{E}_{\mathcal{D}}(L))$ and develop the corresponding calculus of variations over a set of variations denoted by $\mathcal{V}_{\mathcal{D}}$. The characterization of the extremals for such functionals leads to a generalized Euler-Lagrange equation denoted by $(EL)_{\mathcal{D}}$. This procedure is illustrated by the following diagram:

$$\begin{tikzcd} L \arrow[r, "\mathcal{E}_{\mathcal{D}}"] \arrow[d, leftrightarrow, "\mathcal{V}"'] & \mathcal{E}_{\mathcal{D}}(L) \arrow[d, leftrightarrow, "\mathcal{V}_{\mathcal{D}}"'] \\ (EL) & (EL)_{\mathcal{D}} \end{tikzcd}$$

Using these two parts, one can discuss the previous problem to give a more intrinsic origin to a generalized equation. As we have two different ways to generalized the classical Euler-Lagrange equation, via the algebraic or analytic way, a natural question is the following : do we have $\mathcal{E}_{\mathcal{D}}(EL) = (EL)_{\mathcal{D}}$?

The problem can hence be summed up by the following diagram:

$$\begin{tikzcd} L \arrow[r, "\mathcal{E}_{\mathcal{D}}"] \arrow[d, leftrightarrow, "\mathcal{V}"'] & \mathcal{E}_{\mathcal{D}}(L) \arrow[d, leftrightarrow, dotted, "\mathcal{V}_{\mathcal{D}}"', "?"] \\ (EL) \arrow[r, "\mathcal{E}_{\mathcal{D}}"] & \mathcal{E}_{\mathcal{D}}(EL) \end{tikzcd} \tag{1.2}$$

A natural notion with respect to a given generalization associated to a fixed embedding is then the following :

***Definition 1* (Coherence).** — *The embedding $\mathcal{E}_{\mathcal{D}}$ is said to be* coherent *if diagram* (1.2) *is valid, i.e. if the extremals of $\mathcal{A}(\mathcal{E}_{\mathcal{D}}(L))$ according to the variations set $\mathcal{V}_{\mathcal{D}}$ are exactly the solutions of $\mathcal{E}_{\mathcal{D}}(EL)$.*

1.4. A notion of fractional embedding. — In the fractional framework, $\mathcal{D}$ is chosen as a fractional derivative of order α, with $0 < \alpha < 1$. Several definitions exist, and we refer to [**32, 19, 26, 28**] for a detailed presentation of the fractional calculus.

Two types of fractional derivatives exist: the left and the right ones, which respectively involve left and right values of the function. For instance, if $\mathcal{D}^\alpha f(t)$ is a left fractional derivative of order α of f, evaluated in t, it depends on $f(\tau)$, with $\tau < t$.

If a differential equation has a physical content, it should only involve left derivatives. Indeed, the state of a system at time t should be fixed by its past states at times τ, $\tau < t$. We also note that if we study the reversibility of a system, equations describing the backward evolution should only contain right derivatives [**13**]. This motivates the following definition.

***Definition 2* (Causality).** — *A fractional differential equation is said to be* causal *if it involves fractional derivatives of a single type.*

A fractional embedding $\mathcal{E}_\alpha$ of Lagrangian systems has been presented in [**8**]. Similar diagrams to (1.2) have been obtained. However, they are neither coherent nor causal because of the asymmetry of the formula for fractional integration by parts, which makes left and right fractional derivatives appear. If no restrictions are done on variations, the Euler-Lagrange equation derived from the embedded Lagrangian $\mathcal{E}_\alpha(L)$ (denoted here by $(EL)_\alpha$) is different from the embedded Euler-Lagrange equation $\mathcal{E}_\alpha(EL)$ (see diagram (1.3)).

$$\begin{tikzcd} L \arrow[rr, "\mathcal{E}_\alpha"] \arrow[d, leftrightarrow, "\mathcal{V}"'] & & \mathcal{E}_\alpha(L) \arrow[d, leftrightarrow] \\ (EL) \arrow[r, "\mathcal{E}_\alpha"] & \mathcal{E}_\alpha(EL) \neq & (EL)_\alpha \end{tikzcd} \tag{1.3}$$

It has also been shown that by restricting the space of variations (denoted here by $\mathcal{V}_\alpha$), the solutions of $\mathcal{E}_\alpha(EL)$ are some extremals of $\mathcal{A}(\mathcal{E}_{\mathcal{D}}(L))$.

Furthermore, in [**8, 9**], some classical dissipative equations, such as the linear friction equation

$$m\frac{d^2}{dt^2}x(t) + \gamma\frac{d}{dt}x(t) - U'(x(t)) = 0, \tag{1.4}$$

and the diffusion equation

$$\frac{\partial}{\partial t}u(x,t) = c\,\Delta u(x,t) \tag{1.5}$$

have been identified with embedded Euler-Lagrange equations, with a fractional derivative of order $\alpha = 1/2$.

However, there is no equivalence between solutions of the differential equations and extremal points: an extremal point may not be a solution of the differential equation (see diagram (1.6)). Precisions have been given in [**14**], but they have not led to a strict equivalence.

$$\begin{tikzcd} L \arrow[r, "\mathcal{E}_\alpha"] \arrow[d, leftrightarrow, "\mathcal{V}"'] & \mathcal{E}_\alpha(L) \\ (EL) \arrow[r, "\mathcal{E}_\alpha"] & \mathcal{E}_\alpha(EL) \arrow[u, "\mathcal{V}_\alpha"'] \end{tikzcd} \tag{1.6}$$

1.5. Main results. — In this article, we generalize the fractional embedding developed in [**8**], by splitting in two the functional space of the trajectories. The aim is to take into account the asymmetry between left and right fractional derivatives. Our main results are contained in (Theorem 4 and Corollaries 3 and 4) which tell that this *asymmetric fractional embedding* is both coherent and causal solving a classical problem in the fractional calculus of variations.

We then prove (see Theorems 7 and 8) that solutions of equations (1.4) and (1.5) are exactly the extremal points of some asymmetric fractional embeddings of Lagrangian systems. The key point is causality, which allows to see d/dt as $\mathcal{D}^{1/2} \circ \mathcal{D}^{1/2}$. Other fractional Euler-Lagrange equations have been presented in [**29, 1, 5, 15**]. However, they cannot provide a term in d/dt: compositions of fractional derivatives are always between right and left ones.

This article continues with a brief presentation of Lagrangian systems in section 2. Then the asymmetric fractional embedding, presented in Section 3, is proved to be coherent and causal in Section 4. Generalizations, given in Section 5, provide some applications for classical dissipative equations in Section 6. Those results are discussed in Section 7, while some proofs are given in a last Section 8.

2. Lagrangian systems

First of all, we present some basic tools: some functional spaces, the calculus of variations used here for the least action principle and a brief overview on Lagrangian systems.

2.1. Functional spaces. — For two sets A and B, we denote the vector space of functions $f : A \to B$ by $\mathcal{F}(A, B)$. Let $a, b \in \mathbb{R}$, $a < b$. Let $m, n \in \mathbb{N}^*$ and $p \in \mathbb{N}$. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^m$ or the finite interval $[a, b]$. The vector space of functions $\mathcal{U} \to \mathbb{R}^n$ of class C^p is denoted by $C^p(\mathcal{U})$. If Ω is an open subset of $\mathbb{R}^m$, we set

$$\begin{aligned} C^p(\Omega \times [a,b]) = \{\, f \in \mathcal{F}(\Omega \times [a,b]) \mid \forall\, t \in [a,b],\, x \mapsto f(x,t) \in C^p(\Omega), \\ \forall\, x \in \Omega,\, t \mapsto f(x,t) \in C^p([a,b]) \,\} \,. \end{aligned}$$

For $p = 0$, we introduce the following vector spaces:

$$C^0_+([a,b]) = \{f \in C^0([a,b]) \mid f(a) = 0\},$$

$$C^0_-([a,b]) = \{f \in C^0([a,b]) \mid f(b) = 0\},$$

and for $p \geq 1$, we set

$$C_+^p([a,b]) = \{f \in C^p([a,b]) \mid f^{(k)}(a) = 0,\ 0 \le k \le p-1\},$$

$$C_-^p([a,b]) = \{f \in C^p([a,b]) \mid f^{(k)}(b) = 0,\ 0 \le k \le p-1\}.$$

Moreover, for all $p \in \mathbb{N}$, the intersection of $C_+^p([a,b])$ and $C_-^p([a,b])$ is denoted by $C_0^p([a,b])$.

We also denote the set of absolutely continuous functions on $[a,b]$ by $AC([a,b])$, and the set of functions f which have continuous derivatives up to order $(p-1)$ with $f^{(p-1)} \in AC([a,b])$ by $AC^p([a,b])$. The inclusion $C^p([a,b]) \subset AC^p([a,b])$ is obvious.

2.2. Calculus of variations. — As mentioned above, we are interested in the extremals of the Lagrangian action, according to specified variations. These are at the origin of the following definitions, mainly taken from [**3, 8**].

Let A be a vector space, B a subspace of A, and $f : A \to \mathbb{R}$ a function.

Definition 3. — *Let $x \in A$. The function f has a B-minimum (respectively B-maximum) point at x if for all $h \in B$, $f(x+h) \ge f(x)$ (respectively $f(x+h) \le f(x)$).*

The function f has a B-extremum point at x if it has a B-minimum point or a B-maximum point at x.

Definition 4. — *Let $x \in A$. The function f is B-differentiable at x if*

$$f(x+\varepsilon h) = f(x) + \varepsilon df(x,h) + o(\varepsilon),$$

for all $h \in B$, $\varepsilon > 0$, where $h \mapsto df(x,h)$ is a linear function.

Definition 5. — *Let $x \in A$. We suppose that f is B-differentiable at x. The point x is a B-extremal for f if for all $h \in B$, $df(x,h) = 0$.*

In the differentiable case, the classical necessary condition remains with those definitions.

Lemma 1. — *Let $x \in A$. We suppose that f is B-differentiable at x. If f has a B-extremum point at $x \in A$, then x is a B-extremal for f.*

Proof. — We consider the case of a B-minimum.

Let $h \in B$. For all $\varepsilon > 0$, $\varepsilon h \in B$, and $f(x+\varepsilon h) \ge f(x)$. Consequently, $df(x,h) \ge 0$. Moreover, for all $\varepsilon > 0$, $-\varepsilon h \in B$, so $f(x+\varepsilon(-h)) = f(x-\varepsilon h) \ge f(x)$. Then $df(x,-h) \ge 0$. Since $df(x,-h) = -df(x,h)$, we conclude that $df(x,h) = 0$. □

Remark 1. — *The converse is false. However, the misnomer "extremal" for definition 5 may be explained by the fact that necessary condition $df(x,h) = 0$ is widely used for optimization problems.*

2.3. Lagrangian systems. — We give here a short presentation of classical Lagrangian systems by using the previous definitions. By doing so, the understanding of the asymmetric fractional embedding will be made easier. A detailed presentation of Lagrangian systems may be found in [**3, 2**].

Lagrangian systems are totally determined by a single function, the Lagrangian. In this article, we define a Lagrangian as follows.

Definition 6. — *A Lagrangian is a function*

$$\begin{array}{rccc} L: & \mathbb{R}^{2n}\times[a,b] & \longrightarrow & \mathbb{R} \\ & (x,v,t) & \longmapsto & L(x,v,t) \end{array}$$

which verifies the following properties:
- $L \in C^1(\mathbb{R}^{2n}\times[a,b])$,
- $\forall t \in [a,b],\ \forall x \in \mathbb{R}^n,\ v \mapsto L(x,v,t) \in C^2(\mathbb{R}^n)$.

From now on, we denote by $\partial_1 L(x,v,t)$ and $\partial_2 L(x,v,t)$ the differentials of L according respectively to x and v. Hence $\partial_1 L(x,v,t)$ and $\partial_2 L(x,v,t)$ are $\mathbb{R}^n$ valued vectors.

A Lagrangian defines a functional on $C^1([a,b])$, denoted by

$$\begin{array}{rccc} \mathcal{A}(L): & C^1([a,b]) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \displaystyle\int_a^b L\left(x(t),\frac{d}{dt}x(t),t\right)\,dt, \end{array} \tag{2.1}$$

and called the *Lagrangian action*.

According to the least action principle, trajectories of the system are given by the extrema of this action. Using lemma 1, we are interested in extremals for $\mathcal{A}(L)$.

Lemma 2. — *Let L be a Lagrangian and $x \in C^1([a,b])$.*

We suppose that $t \mapsto \partial_2 L(x(t), \frac{d}{dt}x(t), t) \in AC([a,b])$.

Then $\mathcal{A}(L)$ is $C_0^1([a,b])$-differentiable at x and for all $h \in C_0^1([a,b])$,

$$d\mathcal{A}(L)(x,h) = \int_a^b \left[\partial_1 L\left(x(t),\frac{d}{dt}x(t),t\right) - \frac{d}{dt}\partial_2 L\left(x(t),\frac{d}{dt}x(t),t\right)\right]\cdot h(t)\,dt.$$

Proof. — Let $h \in C_0^1([a,b])$ and $\varepsilon > 0$. For all $t \in [a,b]$,

$$\begin{aligned} L\left(x(t)+\varepsilon h(t),\frac{d}{dt}x(t)+\varepsilon\frac{d}{dt}h(t),t\right) &= L\left(x(t),\frac{d}{dt}x(t),t\right) \\ &+\varepsilon\left[\partial_1 L\left(x(t),\frac{d}{dt}x(t),t\right)\cdot h(t) + \partial_2 L\left(x(t),\frac{d}{dt}x(t),t\right)\cdot\frac{d}{dt}h(t)\right] + o(\varepsilon). \end{aligned}$$

Hence we obtain

$$\begin{aligned} \mathcal{A}(L)(x+\varepsilon h) &= \int_a^b L\left(x(t)+\varepsilon h(t),\frac{d}{dt}x(t)+\varepsilon\frac{d}{dt}h(t),t\right)\,dt \\ &= \mathcal{A}(L)(x) + \varepsilon\int_a^b \left[\partial_1 L\left(x(t),\frac{d}{dt}x(t),t\right)\cdot h(t) + \partial_2 L\left(x(t),\frac{d}{dt}x(t),t\right)\cdot\frac{d}{dt}h(t)\right]dt + o(\varepsilon). \end{aligned}$$

Furthermore, since $t \mapsto \partial_2 L\left(x(t),\frac{d}{dt}x(t),t\right) \in AC([a,b])$, an integration by parts gives

$$\int_a^b \partial_2 L\left(x(t),\frac{d}{dt}x(t),t\right)\cdot\frac{d}{dt}h(t)\,dt = -\int_a^b \frac{d}{dt}\partial_2 L\left(x(t),\frac{d}{dt}x(t),t\right)\cdot h(t)\,dt.$$

No boundary term appears because $h \in C_0^1([a,b])$.

Finally, $h \mapsto \displaystyle\int_a^b \left[\partial_1 L\left(x(t),\frac{d}{dt}x(t),t\right) - \frac{d}{dt}\partial_2 L\left(x(t),\frac{d}{dt}x(t),t\right)\right]\cdot h(t)\,dt$ is linear, which concludes the proof.

□

Hence we obtain the following characterization for the $C_0^1([a,b])$-extremal.

Theorem 1. — *Let L be a Lagrangian and $x \in C^1([a,b])$.*
We suppose that $t \mapsto \partial_2 L(x(t), \frac{d}{dt}x(t), t) \in AC([a,b])$. Then we have the following equivalence:
x is a $C_0^1([a,b])$-extremal for $\mathcal{A}(L)$ if and only if x verifies the Euler-Lagrange equation

$$(EL) \qquad \forall t \in [a,b], \quad \partial_1 L\left(x(t), \frac{d}{dt}x(t), t\right) - \frac{d}{dt}\partial_2 L\left(x(t), \frac{d}{dt}x(t), t\right) = 0. \tag{2.2}$$

Proof. — From Lemma 2, x is a $C_0^1([a,b])$-extremal for $\mathcal{A}(L)$ if and only if for all $h \in C_0^1([a,b])$,

$$\int_a^b \left[\partial_1 L\left(x(t), \frac{d}{dt}x(t), t\right) - \frac{d}{dt}\partial_2 L\left(x(t), \frac{d}{dt}x(t), t\right)\right] \cdot h(t)\, dt = 0.$$

We conclude by using the fundamental lemma in the calculus of variations [**2**, p.57].

□

3. Asymmetric fractional embedding

We begin with a brief presentation of the fractional operators which will be used in this article. Then we present the asymmetric fractional embedding as in [**8**]: firstly for differential operators and secondly for Lagrangian systems.

3.1. Fractional operators. —

3.1.1. Fractional integrals. — Let $\beta > 0$, and $f, g : [a,b] \to \mathbb{R}^n$.

Definition 7. — *The left and right Riemann-Liouville fractional integrals are respectively defined by*

$$\mathcal{I}_+^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-\tau)^{\beta-1} f(\tau)\, d\tau,$$

$$\mathcal{I}_-^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_t^b (\tau-t)^{\beta-1} f(\tau)\, d\tau,$$

for $t \in [a,b]$, where Γ is the gamma function.

Lemma 3. — 1. *If $f \in AC([a,b])$, then $\mathcal{I}_+^\beta f \in AC([a,b])$ and $\mathcal{I}_-^\beta f \in AC([a,b])$.*
2. *If $f \in C^0([a,b])$, then $\mathcal{I}_+^\beta f \in C_+^0([a,b])$ and $\mathcal{I}_-^\beta f \in C_-^0([a,b])$.*

Proof. — 1. We refer to Lemma 2.1 in [**32**, p.32].
2. We refer to Theorem 3.1 in [**32**, p.53], with $\lambda = 0$.

□

Composing those fractional integrals with the usual derivative lead to the following fractional derivatives.

3.1.2. Fractional derivatives. — Let $\alpha > 0$. Let $p \in \mathbb{N}$ such that $p-1 \leq \alpha < p$.

Definition 8. — *The left and right Riemann-Liouville fractional derivatives are respectively defined by*

$$\begin{aligned}
\mathcal{D}^{\alpha}_{+} f(t) &= \left(\frac{d^p}{dt^p} \circ \mathcal{I}^{p-\alpha}_{+} \right) f(t) \\
&= \frac{1}{\Gamma(p-\alpha)} \frac{d^p}{dt^p} \int_a^t (t-\tau)^{p-1-\alpha} f(\tau)\, d\tau,
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^{\alpha}_{-} f(t) &= \left((-1)^p \frac{d^p}{dt^p} \circ \mathcal{I}^{p-\alpha}_{-} \right) f(t) \\
&= \frac{(-1)^p}{\Gamma(p-\alpha)} \frac{d^p}{dt^p} \int_t^b (\tau-t)^{p-1-\alpha} f(\tau)\, d\tau,
\end{aligned}$$

for $t \in [a,b]$.

If we change the order of composition, we obtain another definition.

Definition 9. — *The left and right Caputo fractional derivatives are respectively defined by*

$$\begin{aligned}
{}^c\mathcal{D}^{\alpha}_{+} f(t) &= \left(\mathcal{I}^{p-\alpha}_{+} \circ \frac{d^p}{dt^p} \right) f(t) \\
&= \frac{1}{\Gamma(p-\alpha)} \int_a^t (t-\tau)^{p-1-\alpha} f^{(p)}(\tau)\, d\tau,
\end{aligned}$$

$$\begin{aligned}
{}^c\mathcal{D}^{\alpha}_{-} f(t) &= \left(\mathcal{I}^{p-\alpha}_{-} \circ (-1)^p \frac{d^p}{dt^p} \right) f(t) \\
&= \frac{(-1)^p}{\Gamma(p-\alpha)} \int_t^b (\tau-t)^{p-1-\alpha} f^{(p)}(\tau)\, d\tau,
\end{aligned}$$

for $t \in [a,b]$.

The link between Riemann-Liouville and Caputo derivatives is given by Theorem 2.2 in [**32**, p.39]:

Theorem 2. — *For* $f \in AC^p([a,b])$, $\mathcal{D}^{\alpha}_{+} f$ *exists almost everywhere, and for all* $t \in (a,b]$,

$$\mathcal{D}^{\alpha}_{+} f(t) = \sum_{k=0}^{p-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)} (t-a)^{k-\alpha} + {}^c\mathcal{D}^{\alpha}_{+} f(t).$$

Lemma 3 and Theorem 2 directly provide the following results.

Lemma 4. — 1. *If* $f \in C^p([a,b])$, *then* ${}^c\mathcal{D}^{\alpha}_{+} f \in C^0_{+}([a,b])$.

2. *If* $f \in C^p_{+}([a,b])$, *then* $\mathcal{D}^{\alpha}_{+} f = {}^c\mathcal{D}^{\alpha}_{+} f$. *In particular,* $\mathcal{D}^{\alpha}_{+} f \in C^0_{+}([a,b])$.

Similar results hold for the right derivatives.

3.2. Asymmetric fractional embedding of differential operators. — We adapt here the presentation done in [**8**].

Let $M, N \in \mathbb{N}^*$. If $f \in \mathcal{F}(\mathbb{R}^{M+1}, \mathbb{R}^N)$ and $y \in \mathcal{F}([a,b], \mathbb{R}^M)$, we denote by $f(y(\bullet), \bullet)$ the function defined by

$$\begin{array}{rccl} f(y(\bullet),\bullet): & [a,b] & \longrightarrow & \mathbb{R}^N \\ & t & \longmapsto & f(y(t),t). \end{array}$$

Let $p, k \in \mathbb{N}$. If $\mathbf{f} = \{f_i\}_{0\le i\le p}$ and $\mathbf{g} = \{g_j\}_{1\le j\le p}$ are two families of $\mathcal{F}(\mathbb{R}^{n(k+1)+1}, \mathbb{R}^m)$ ($\mathbf{g} = \emptyset$ if $p = 0$), with $f_j \in C^0(\mathbb{R}^{n(k+1)+1})$ and $g_j \in C^j(\mathbb{R}^{n(k+1)+1})$ for $1 \le j \le p$, we introduce the operator $\mathcal{O}_{\mathbf{f}}^{\mathbf{g}}$ defined by

$$\begin{array}{rccl} \mathcal{O}_{\mathbf{f}}^{\mathbf{g}}: & C^{k+p}([a,b]) & \longrightarrow & \mathcal{F}\left([a,b], \mathbb{R}^m\right) \\ & x & \longmapsto & \left[f_0 + \displaystyle\sum_{i=1}^{p} f_i \star \frac{d^i}{dt^i} g_i\right]\left(x(\bullet), \dots, \frac{d^k}{dt^k}x(\bullet), \bullet\right), \end{array} \tag{3.1}$$

where, for two operators $A = (A_1, \dots, A_m)$ and $B = (B_1, \dots, B_m)$, $A \star B$ is defined by

$$(A \star B)(y) = (A_1(y)B_1(y), \dots, A_m(y)B_m(y)).$$

The fractional embedding presented in [**8**] consists in replacing d/dt by a fractional derivative. Here we want to keep this idea, but additionnaly we want to split in two the functional space of the trajectories, in order to make the asymmetry between left and right fractional derivatives explicitly appear.

Let $0 < \alpha < 1$. For $X = (x_+, x_-) \in C^1([a,b])^2$ ($AC^2([a,b])^2$ would be sufficient), we introduce the fractional derivative ${}^c\mathcal{D}^\alpha$, defined by

$${}^c\mathcal{D}^\alpha X = \left({}^c\mathcal{D}^\alpha_+ \, x_+, -\, {}^c\mathcal{D}^\alpha_- \, x_-\right).$$

The classical case is recovered for $\alpha \to 1^-$ (and not for $\alpha = 1$).

Lemma 5. — *Let $X \in C^1([a,b])^2$. Then*

$$\forall\, t \in (a,b), \ \lim_{\alpha\to 1^-} {}^c\mathcal{D}^\alpha X(t) = \frac{d}{dt}X(t).$$

Proof. — See Section 8. □

Hence for $k \in \mathbb{N}^*$ and a "suitable" function X,

$$({}^c\mathcal{D}^\alpha)^k X = \left(({}^c\mathcal{D}^\alpha_+)^k x_+, (-\,{}^c\mathcal{D}^\alpha_-)^k x_-\right).$$

The following lemma provides an example of such "suitable" functions.

Lemma 6. — *Let $k \in \mathbb{N}^*$. If $f \in C^k_+([a,b])$, then we have*

$$\left({}^c\mathcal{D}^\alpha_+\right)^k f = {}^c\mathcal{D}^{\alpha k}_+ f.$$

A similar result holds for the right derivative.

Proof. — See Section 8. □

Consequently, if $X = (x_+, x_-) \in C^k_+([a,b]) \times C^k_-([a,b])$, $({}^c\mathcal{D}^\alpha)^k X$ verifies

$$({}^c\mathcal{D}^\alpha)^k X = ({}^c\mathcal{D}^{\alpha k}_+ \, x_+, (-1)^k \, {}^c\mathcal{D}^{\alpha k}_- \, x_-),$$

and $({}^c\mathcal{D}^\alpha)^k X \in C^0_+([a,b]) \times C^0_-([a,b])$.

Let us now precise the splitting we are interested in.

Definition 10. — *Let $k \in \mathbb{N}$ and $m, n \in \mathbb{N}^*$. Let $f \in \mathcal{F}(\mathbb{R}^{n(k+1)+1}, \mathbb{R}^m)$. The asymmetric representation of f, denoted by $\tilde{f}$, is defined by*

$$\begin{array}{rccc} \tilde{f}: & \mathbb{R}^{2n(k+1)+1} & \longrightarrow & \mathbb{R}^m \\ & (x_0, y_0, \dots, x_k, y_k, t) & \longmapsto & f(x_0 + y_0, \dots, x_k + y_k, t). \end{array}$$

Actually, the relevant functions will be in $\mathcal{F}([a,b], \mathbb{R}^n) \times \{0\}$ or $\{0\} \times \mathcal{F}([a,b], \mathbb{R}^n)$. That is why we introduce the following "selection" matrix.

Let $\mathcal{M}_{m,2m}(\mathbb{R})$ be the set of real matrices with m rows and $2m$ columns. We note I_m the identity matrix of dimension m, and we introduce the operator σ defined by

$$\begin{array}{rccl} \sigma: & \mathcal{F}([a,b], \mathbb{R}^m)^2 & \longrightarrow & \mathcal{M}_{m,2m}(\mathbb{R}) \\ & X & \longmapsto & (I_m \quad 0) \quad \text{if } X \in \mathcal{F}([a,b], \mathbb{R}^n) \times \{0\} \text{ and } X \neq 0, \\ & & & (0 \quad I_m) \quad \text{if } X \in \{0\} \times \mathcal{F}([a,b], \mathbb{R}^n) \text{ and } X \neq 0, \\ & & & (0 \quad 0) \quad \text{otherwise.} \end{array}$$

Now we can define the asymmetric fractional embedding of a differential operator.

Definition 11. — *With the previous notations, the asymmetric fractional embedding of operator (3.1), denoted by $\mathcal{E}_\alpha(\mathcal{O}_{\boldsymbol{f}}^{\boldsymbol{g}})$, is defined on a subset $E^\alpha \subset \mathcal{F}([a,b], \mathbb{R}^n)^2$, by*

$$\begin{array}{rccl} \mathcal{E}_\alpha(\mathcal{O}_{\boldsymbol{f}}^{\boldsymbol{g}}): & E^\alpha & \longrightarrow & \mathcal{F}([a,b], \mathbb{R}^m) \\ & X & \longmapsto & \left[\tilde{f}_0 + \sigma(X) \displaystyle\sum_{i=1}^{p} \begin{pmatrix} \tilde{f}_i \star \mathcal{D}_+^{\alpha i} \tilde{g}_i \\ \tilde{f}_i \star (-1)^i \mathcal{D}_-^{\alpha i} \tilde{g}_i \end{pmatrix} \right] \Big(X(\bullet), \dots, ({}^c\mathcal{D}^\alpha)^k X(\bullet), \bullet \Big). \end{array} \tag{3.2}$$

The definition set E^α of $\mathcal{E}_\alpha(\mathcal{O}_{\mathbf{f}}^{\mathbf{g}})$ depends on $\mathbf{f}$ and $\mathbf{g}$. We also introduce the following spaces:

$$E_+^\alpha = E^\alpha \cap (\mathcal{F}([a,b], \mathbb{R}^n) \times \{0\}), \quad E_-^\alpha = E^\alpha \cap (\{0\} \times \mathcal{F}([a,b], \mathbb{R}^n)).$$

In particular, for $(x_+, 0) \in E_+^\alpha$, (3.2) becomes

$$\mathcal{E}_\alpha(\mathcal{O}_{\mathbf{f}}^{\mathbf{g}})(x_+, 0)(t) = \left[f_0 + \sum_{i=1}^{p} f_i \star \mathcal{D}_+^{\alpha i} g_i \right] \Big(x_+(t), \dots, ({}^c\mathcal{D}_+^\alpha)^k x_+(t), t \Big),$$

and for $(0, x_-) \in E_-^\alpha$, we have

$$\mathcal{E}_\alpha(\mathcal{O}_{\mathbf{f}}^{\mathbf{g}})(0, x_-)(t) = \left[f_0 + \sum_{i=1}^{p} f_i \star (-1)^i \mathcal{D}_-^{\alpha i} g_i \right] \Big(x_-(t), \dots, (-{}^c\mathcal{D}_-^\alpha)^k x_-(t), t \Big).$$

Remark 2. — *There exists of course several ways to define a fractional embedding because of the different definitions of fractional derivatives. As it will be seen later, with our choice, the action of a fractional Lagrangian is well defined, we may obtain a coherent and causal embedding, and the differential equations presented in Section 6 will have relevant solutions.*

For the sake of clarity, we will often denote by $\overline{x}$ the integer which verifies $\overline{x} - 1 \leq x < \overline{x}$, where $x \in \mathbb{R}^+$. We also denote by $\underline{x}$ the integer which verifies $\underline{x} - 1 < x \leq \underline{x}$.

Precisions on E_+^α and E_-^α can be given thanks to the following lemma.

Lemma 7. — *Let $\beta > 0$ and $p \in \mathbb{N}$. If $f \in C_+^{\underline{\beta}+p}([a,b])$, then ${}^c\mathcal{D}_+^\beta f \in C_+^p([a,b])$. A similar result holds for the right derivative.*

Proof. — See section 8. □

Corollary 1. — – *If* $\dfrac{\partial g_i}{\partial t} = 0$ *for all* $1 \leq i \leq p$, $C_+^{p+k}([a,b]) \times C_-^{p+k}([a,b]) \subset E^\alpha$, *and for all* $X \in C_+^{p+k}([a,b]) \times C_-^{p+k}([a,b])$, $\mathcal{E}_\alpha(\mathcal{O}_{\boldsymbol{f}}^{\boldsymbol{g}})(X) \in C^0([a,b])$.

– *If* $p = 0$ *(*$\boldsymbol{g} = \emptyset$*) and* $k = 1$, $C^1([a,b])^2 \subset E^\alpha$ *and for all* $X \in C^1([a,b])^2$, $\mathcal{E}_\alpha(\mathcal{O}_{\boldsymbol{f}}^{\emptyset})(X) \in C^0([a,b])$.

Proof. — – Let $X = (x_+, x_-) \in C_+^{p+k}([a,b]) \times C_-^{p+k}([a,b])$. For all $1 \leq j \leq k$, $({}^c\mathcal{D}^\alpha)^j X = ({}^c\mathcal{D}_+^{\alpha j}\, x_+, (-1)^j\, {}^c\mathcal{D}_-^{\alpha j}\, x_-)$ and $({}^c\mathcal{D}^\alpha)^j X \in C_+^p([a,b]) \times C_-^p([a,b])$, from Lemmas 6 and 7.

If $x_+ \neq 0$ and $x_- \neq 0$, $\mathcal{E}_\alpha(\mathcal{O}_{\mathbf{f}}^{\mathbf{g}})(X) = \tilde{f}_0\left(X(\bullet), \ldots, ({}^c\mathcal{D}^\alpha)^k X(\bullet), \bullet\right) \in C^0([a,b])$.

If $x_- = 0$, let $1 \leq i \leq p$. Since g_i is of class C^i, we have $g_i(x_+) \,:\, t \mapsto g_i(x_+(t), \ldots, {}^c\mathcal{D}_+^{\alpha k}\, x_+(t), t) \in C^i([a,b])$. Moreover, $g_i(x_+)'(a) = \displaystyle\sum_{j=1}^{k} \partial_j g_i(x_+)(a) \cdot x_+^{(j)}(a) + \frac{\partial g_i(x_+)}{\partial t}(a)$. Since $\dfrac{\partial g_i}{\partial t} = 0$ and $x_+^{(j)}(a) = 0$ for all $1 \leq j \leq k$, we obtain $g_i(x_+)'(a) = 0$. By induction, $g_i(x_+)^{(l)}(a) = 0$ for all $1 \leq l \leq i$. Hence $g_i(x_+) \in C_+^i([a,b])$, and from Lemma 4, $\mathcal{D}_+^{\alpha i}\, g_i(x_+) \in C^0([a,b])$.

We proceed likewise if $x_+ = 0$.

– Let $X \in C^1([a,b])^2$. We have $\tilde{f}_0(X(\bullet), {}^c\mathcal{D}^\alpha X(\bullet), \bullet) \in C^0([a,b])$ from Lemma 4, so $\mathcal{E}_\alpha(\mathcal{O}_{\mathbf{f}}^{\emptyset})(X) = \tilde{f}_0(X(\bullet), {}^c\mathcal{D}^\alpha X(\bullet), \bullet)$ is well defined and is a function of $C^0([a,b])$.

□

In order to clarify those notations, we give here a short example.

Example 1. — *We set* $n = m = p = 1$, $k = 2$, *and we suppose that* $0 < \alpha < 1/2$.

Let f_0, f_1, $g_1 \,:\, \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$ *be three functions defined by*

$$\begin{aligned} f_0(a,b,c,t) &= c + e^{-t}\cos b, \\ f_1(a,b,c,t) &= 1, \\ g_1(a,b,c,t) &= \cos a. \end{aligned}$$

The associated operator $\mathcal{O}_{\boldsymbol{f}}^{\boldsymbol{g}}$ *verifies*

$$\mathcal{O}_{\boldsymbol{f}}^{\boldsymbol{g}}(x)(t) = \frac{d^2}{dt^2}x(t) + e^{-t}\cos\left(\frac{d}{dt}x(t)\right) + \frac{d}{dt}\cos(x(t)),$$

for $x \in C^2([a,b])$ *and* $t \in [a,b]$.

Moreover, for any $(x_+, x_-) \in AC^2([a,b])^2$, $({}^c\mathcal{D}^\alpha)^2(x_+, x_-) = ({}^c\mathcal{D}_+^{2\alpha}\, x_+,\, {}^c\mathcal{D}_-^{2\alpha}\, x_-)$ *as it will be shown in Lemma 13. The asymmetric fractional embedding* $\mathcal{E}_\alpha(\mathcal{O}_{\boldsymbol{f}}^{\boldsymbol{g}})$ *is hence given by*

$$\begin{aligned} \mathcal{E}_\alpha(\mathcal{O}_{\boldsymbol{f}}^{\boldsymbol{g}})(x_+, x_-)(t) = {}& {}^c\mathcal{D}_+^{2\alpha}\, x_+(t) + {}^c\mathcal{D}_-^{2\alpha}\, x_-(t) + e^{-t}\cos({}^c\mathcal{D}_+^{\alpha}\, x_+(t) - {}^c\mathcal{D}_-^{\alpha}\, x_-(t)) \\ & + \sigma(x_+, x_-) \begin{pmatrix} \mathcal{D}_+^\alpha\, \cos(x_+(t) + x_-(t)) \\ -\,\mathcal{D}_-^\alpha\, \cos(x_+(t) + x_-(t)) \end{pmatrix}. \end{aligned}$$

For $(x_+, 0) \in AC^2([a,b]) \times \{0\}$, *the fractional embedding becomes*

$$\mathcal{E}_\alpha(\mathcal{O}_{\boldsymbol{f}}^{\boldsymbol{g}})(x_+, 0)(t) = {}^c\mathcal{D}_+^{2\alpha}\, x_+(t) + e^{-t}\cos({}^c\mathcal{D}_+^{\alpha}\, x_+(t)) + \mathcal{D}_+^\alpha\, \cos(x_+(t)),$$

and for $(0, x_-) \in \{0\} \times AC^2([a,b])$, *we have*

$$\mathcal{E}_\alpha(\mathcal{O}_{\boldsymbol{f}}^{\boldsymbol{g}})(0, x_-)(t) = {}^c\mathcal{D}_-^{2\alpha}\, x_-(t) + e^{-t}\cos(-\,{}^c\mathcal{D}_-^{\alpha}\, x_-(t)) - \mathcal{D}_-^\alpha\, \cos(x_-(t)).$$

The ordinary differential equations may be written by using operators $\mathcal{O}_{\mathbf{f}}^{\mathbf{g}}$. Following [**8**], we consider the differential equations of the form

$$\mathcal{O}_{\mathbf{f}}^{\mathbf{g}}(x) = 0, \quad x \in C^{p+k}([a, b]). \tag{3.3}$$

Definition 12. — *With the previous notations, the asymmetric fractional embedding of differential equation* (3.3) *is defined by*

$$\mathcal{E}_\alpha(\mathcal{O}_{\boldsymbol{f}}^{\boldsymbol{g}})(X) = 0, \quad X \in E^\alpha. \tag{3.4}$$

Consequently, if $(x_+, 0) \in E_+^\alpha$, (3.4) becomes

$$\left[f_0 + \sum_{i=1}^{p} f_i \star \mathcal{D}_+^{\alpha i} g_i \right] \Big(x_+(t), \ldots, ({}^c\mathcal{D}_+^\alpha)^k x_+(t), t \Big) = 0,$$

and for $(0, x_-) \in E_-^\alpha$, we obtain

$$\left[f_0 + \sum_{i=1}^{p} f_i \star (-1)^i \mathcal{D}_-^{\alpha i} g_i \right] \Big(x_-(t), \ldots, ({}^c\mathcal{D}_-^\alpha)^k x_-(t), t \Big) = 0.$$

We verify that for these two cases, the asymmetric fractional embedding respects causality, in the sense of Definition 2.

This method is now applied to Lagrangian systems.

3.3. Asymmetric fractional embedding of Lagrangian systems. — Let $0 < \alpha < 1$. Let L be a Lagrangian.

For $X = (x_1, x_2) \in \mathbb{R}^{2n}$, $Y = (y_1, y_2) \in \mathbb{R}^{2n}$, and $t \in [a, b]$, the asymmetric representation of L, denoted by $\tilde{L}$, verifies

$$\tilde{L}(X, Y, t) = L(x_1 + x_2, y_1 + y_2, t).$$

Given that

$$\frac{\partial L}{\partial x_1}(x_1 + x_2, y_1 + y_2, t) = \frac{\partial L}{\partial x_2}(x_1 + x_2, y_1 + y_2, t) = \partial_1 L(x_1 + x_2, y_1 + y_2, t),$$

we deduce $\partial_1 \tilde{L}(X, Y, t) = \partial_1 L(x_1 + x_2, y_1 + y_2, t)$. Similarly, we note $\partial_2 \tilde{L}(X, Y, t) = \partial_2 L(x_1 + x_2, y_1 + y_2, t)$.

Theorem 3. — *The asymmetric fractional embedding of* (2.2) *is defined by*

$$\mathcal{E}_\alpha(EL) \qquad \partial_1 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t) - \sigma(X) \begin{pmatrix} \mathcal{D}_+^\alpha \, \partial_2 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t) \\ -\mathcal{D}_-^\alpha \, \partial_2 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t) \end{pmatrix} = 0. \tag{3.5}$$

In particular, for $(x_+, 0) \in E_+^\alpha$, (3.5) *becomes*

$$\mathcal{E}_\alpha(EL)_+ \qquad \partial_1 L(x_+(t), {}^c\mathcal{D}_+^\alpha x_+(t), t) - \mathcal{D}_+^\alpha \, \partial_2 L(x_+(t), {}^c\mathcal{D}_+^\alpha x_+(t), t) = 0, \tag{3.6}$$

and for $(0, x_-) \in E_-^\alpha$,

$$\mathcal{E}_\alpha(EL)_- \qquad \partial_1 L(x_-(t), -{}^c\mathcal{D}_-^\alpha x_-(t), t) + \mathcal{D}_-^\alpha \, \partial_2 L(x_-(t), -{}^c\mathcal{D}_-^\alpha x_-(t), t) = 0. \tag{3.7}$$

Proof. — Equation (2.2) may be written like (3.3) with $k = 1$, $p = 1$, $\mathbf{f} = \{\partial_1 L, 1\}$ and $\mathbf{g} = \{-\partial_2 L\}$. We conclude by using Definitions 11 and 12. □

On the other hand, the asymmetric fractional embedding of the Lagrangian L, which will be noted L_α, verifies

$$L_\alpha(X)(t) = L(x_+(t) + x_-(t),\, {}^c\mathcal{D}^\alpha_+ x_+(t) - {}^c\mathcal{D}^\alpha_- x_-(t), t),$$

for all $(x_+, x_-) \in C^1([a,b])^2$ and $t \in [a,b]$.

The associated action (2.1) now becomes

$$\begin{array}{rccc} \mathcal{A}(L_\alpha) : & C^1([a,b])^2 & \longrightarrow & \mathbb{R} \\ & X & \longmapsto & \displaystyle\int_a^b \tilde{L}\left(X(t),\, {}^c\mathcal{D}^\alpha X(t), t\right)\, dt. \end{array}$$

Remark 3. — *We see here the necessity to choose the Caputo derivative inside the functions. If we had taken the Riemann-Liouville derivative, the action could be undefined even for regular functions. For example, if* $L(x,v,t) = \dfrac{1}{2}v^2 - U(x)$ *and* $x_+ \in C^1([a,b])$*, with* $x_+(a) \neq 0$*, we would have*

$$L(x_+(t),\, \mathcal{D}^\alpha_+ x_+(t), t) \underset{a}{\sim} \frac{1}{2}\left(\frac{x_+(a)}{\Gamma(1-\alpha)}\right)^2 (t-a)^{-2\alpha},$$

and $\mathcal{A}(L_\alpha)(x_+, 0)$ *would not be defined for* $\alpha \geq 1/2$.

The obtention of the differential of the action first requires a formula for integration by parts with fractional derivatives.

Lemma 8. — *Let* $\beta > 0$*. If* $f \in AC^{\beta}_{-}([a,b])$ *and* $g \in C^{\beta}_0([a,b])$*, then we have the following formula for fractional integration by parts:*

$$\int_a^b f(t) \cdot {}^c\mathcal{D}^\beta_- g(t)\, dt = \int_a^b \mathcal{D}^\beta_+ f(t) \cdot g(t)\, dt.$$

Similarly, we have:

$$\int_a^b f(t) \cdot {}^c\mathcal{D}^\beta_+ g(t)\, dt = \int_a^b \mathcal{D}^\beta_- f(t) \cdot g(t)\, dt.$$

Proof. — See Section 8. □

Lemma 9. — *Let* $X \in C^1([a,b])^2$*. We suppose that* $\partial_2 \tilde{L}(X(\bullet),\, {}^c\mathcal{D}^\alpha X(\bullet), \bullet) \in AC([a,b])$*.*

Then $\mathcal{A}(L_\alpha)$ *is* $C^1_0([a,b])^2$*-differentiable at* X *and for all* $H = (h_+, h_-) \in C^1_0([a,b])^2$*,*

$$\begin{aligned} d\mathcal{A}(L_\alpha)(X,H) = &\int_a^b \left[\partial_1 \tilde{L}\left(X(t),\, {}^c\mathcal{D}^\alpha X(t), t\right) + \mathcal{D}^\alpha_- \partial_2 \tilde{L}\left(X(t),\, {}^c\mathcal{D}^\alpha X(t), t\right)\right] \cdot h_+(t)\, dt \\ &+ \int_a^b \left[\partial_1 \tilde{L}\left(X(t),\, {}^c\mathcal{D}^\alpha X(t), t\right) - \mathcal{D}^\alpha_+ \partial_2 \tilde{L}\left(X(t),\, {}^c\mathcal{D}^\alpha X(t), t\right)\right] \cdot h_-(t)\, dt. \end{aligned}$$

Proof. — Let $H = (h_+, h_-) \in C^1_0([a,b])^2$ and $\varepsilon > 0$. For all $t \in [a,b]$, we have:

$$\begin{aligned} \tilde{L}(X(t) + \varepsilon H(t),\, {}^c\mathcal{D}^\alpha X(t) + \varepsilon\, {}^c\mathcal{D}^\alpha H(t), t) = &\tilde{L}(X(t),\, {}^c\mathcal{D}^\alpha X(t), t) \\ &+ \partial_1 \tilde{L}(X(t),\, {}^c\mathcal{D}^\alpha X(t), t) \cdot (h_+(t) + h_-(t)) \\ &+ \partial_2 \tilde{L}(X(t),\, {}^c\mathcal{D}^\alpha X(t), t) \cdot (\,{}^c\mathcal{D}^\alpha_+ h_+(t) - {}^c\mathcal{D}^\alpha_- h_-(t)) + o(\varepsilon). \end{aligned}$$

Integrating this relation on $[a,b]$ leads to:

$$\mathcal{A}(L_\alpha)(X+\varepsilon H) = \mathcal{A}(L_\alpha)(X) + \varepsilon \int_a^b \partial_1 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t)\cdot (h_+(t)+h_-(t))\,dt$$
$$+\,\varepsilon \int_a^b \partial_2 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t)\cdot ({}^c\mathcal{D}^\alpha_+\, h_+(t) - {}^c\mathcal{D}^\alpha_-\, h_-(t))\,dt + o(\varepsilon).$$

Since $\partial_2 \tilde{L}(X(\bullet), \mathcal{D}X(\bullet), \bullet) \in AC([a,b])$, $h_+ \in C^1_0([a,b])$ and $h_- \in C^1_0([a,b])$, we can use lemma 8:

$$\int_a^b \partial_2 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t)\cdot {}^c\mathcal{D}^\alpha_+\, h_+(t)\,dt = \int_a^b \mathcal{D}^\alpha_-\, \partial_2 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t)\cdot h_+(t)\,dt,$$

and

$$\int_a^b \partial_2 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t)\cdot {}^c\mathcal{D}^\alpha_-\, h_-(t)\,dt = \int_a^b \mathcal{D}^\alpha_+\, \partial_2 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t)\cdot h_-(t)\,dt.$$

Finally, $(h_+, h_-) \mapsto \int_a^b \left[\partial_1 \tilde{L}\,(X(t), {}^c\mathcal{D}^\alpha X(t), t) + \mathcal{D}^\alpha_-\, \partial_2 L\,(X(t), {}^c\mathcal{D}^\alpha X(t), t)\right]\cdot h_+(t)\,dt +$ $\int_a^b \left[\partial_1 \tilde{L}\,(X(t), {}^c\mathcal{D}^\alpha X(t), t) - \mathcal{D}^\alpha_+\, \partial_2 L\,(X(t), {}^c\mathcal{D}^\alpha X(t), t)\right]\cdot h_-(t)\,dt$ is linear, which concludes the proof.

□

Then we obtain a result similar to Theorem 1.

Theorem 4. — *Let $X \in C^1([a,b])^2$. We suppose that $\partial_2 \tilde{L}(X(\bullet), {}^c\mathcal{D}^\alpha X(\bullet), \bullet) \in AC([a,b])$.*
Then we have the following equivalence:
X is a $C^1_0([a,b])^2$-extremal of the action $\mathcal{A}(L_\alpha)$ if and only if it verifies

$$(EL_\alpha) \qquad \forall\, t \in (a,b), \quad \left\{ \begin{array}{lcl} \partial_1 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t) + \mathcal{D}^\alpha_-\, \partial_2 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t) &=& 0, \\ \partial_1 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t) - \mathcal{D}^\alpha_+\, \partial_2 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t) &=& 0. \end{array} \right. \tag{3.8}$$

Proof. — Similar to Theorem 1. The only difference is that $\mathcal{D}^\alpha_-\, \partial_2 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t)$ and $\mathcal{D}^\alpha_+\, \partial_2 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t)$ may not be continuous respectively in b and a. □

Equation (3.8) is very restrictive since X must verify

$$(\,\mathcal{D}^\alpha_+ + \mathcal{D}^\alpha_-\,)\partial_2 \tilde{L}(X(t), {}^c\mathcal{D}^\alpha X(t), t) = 0.$$

This condition may not be related to the dynamics of the system and seems too strong. For instance, for $\alpha \in (1,2)$, functions which fulfill $(\,\mathcal{D}^\alpha_+ + \mathcal{D}^\alpha_-\,)f = 0$ are given in **[22]** and are very specific. By restricting the set of variations, equations more relevant will now be obtained.

4. Coherence and causality

We will see here the interest of the asymmetric fractional embedding.

Euler-Lagrange equations which have been obtained so far in **[29, 1, 8, 15]** involve both left and right fractional derivatives. The following result provides a similar equation.

Corollary 2. — *Let $x_+ \in C^1([a,b])$. We suppose that $\partial_2 L(x_+(\bullet), {}^c\mathcal{D}^\alpha_+ x_+(\bullet), \bullet) \in AC([a,b])$. Then we have the following equivalence:*

$(x_+, 0)$ is a $C^1_0([a,b]) \times \{0\}$-extremal of the action $\mathcal{A}(L_\alpha)$ if and only if x_+ verifies

$$\partial_1 L(x_+(t), {}^c\mathcal{D}^\alpha_+ x_+(t), t) + \mathcal{D}^\alpha_- \, \partial_2 L(x_+(t), {}^c\mathcal{D}^\alpha_+ x_+(t), t) = 0, \tag{4.1}$$

for all $t \in [a,b)$.

Such an equation is not causal because of the simultaneous presence of ${}^c\mathcal{D}^\alpha_+$ and $\mathcal{D}^\alpha_-$. Moreover, regarding (3.6), this procedure is not coherent. Those problems are solved with the following results.

Corollary 3. — *Let $x_+ \in C^1([a,b])$. We suppose that $\partial_2 L(x_+(\bullet), {}^c\mathcal{D}^\alpha_+ x_+(\bullet), \bullet) \in AC([a,b])$. Then we have the following equivalence:*

$(x_+, 0)$ is a $\{0\} \times C^1_0([a,b])$-extremal of the action $\mathcal{A}(L_\alpha)$ if and only if x_+ verifies

$$(EL_\alpha)_+ \qquad \forall\, t \in (a,b], \quad \partial_1 L(x_+(t), {}^c\mathcal{D}^\alpha_+ x_+(t), t) - \mathcal{D}^\alpha_+ \, \partial_2 L(x_+(t), {}^c\mathcal{D}^\alpha_+ x_+(t), t) = 0. \tag{4.2}$$

Corollary 4. — *Let $x_- \in C^1([a,b])$. We suppose that $\partial_2 L(x_-(\bullet), {}^c\mathcal{D}^\alpha_- x_-(\bullet), \bullet) \in AC([a,b])$. Then we have the following equivalence:*

$(0, x_-)$ is a $C^1_0([a,b]) \times \{0\}$-extremal of the action $\mathcal{A}(L_\alpha)$ if and only if x_- verifies

$$(EL_\alpha)_- \qquad \forall\, t \in [a,b), \quad \partial_1 L(x_-(t), {}^c\mathcal{D}^\alpha_- x_-(t), t) - \mathcal{D}^\alpha_- \, \partial_2 L(x_-(t), {}^c\mathcal{D}^\alpha_- x_-(t), t) = 0. \tag{4.3}$$

Equations (4.2) and (4.3) are causal. Moreover, they are respectively similar to (3.6) and (3.7): $(EL_\alpha)_\pm \equiv \mathcal{E}_\alpha(EL)_\pm$. With such sets of variations, the asymmetric fractional embedding is therefore coherent. To sum up, if we note $\mathcal{E}^\pm_\alpha$ the asymmetric fractional embeddings evaluated on trajectories in $E^\alpha_\pm$, the following diagrams are valid:

$$\begin{tikzcd} L \arrow[r, "\mathcal{E}^+_\alpha"] \arrow[d, leftrightarrow, "C^1_0([a,b])"'] & L_\alpha \arrow[d, leftrightarrow, "\{0\}\times C^1_0([a,b])"] \\ (EL) \arrow[r, "\mathcal{E}^+_\alpha"] & \mathcal{E}_\alpha(EL)_+ \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, "\mathcal{E}^-_\alpha"] \arrow[d, leftrightarrow, "C^1_0([a,b])"'] & L_\alpha \arrow[d, leftrightarrow, "C^1_0([a,b])\times\{0\}"] \\ (EL) \arrow[r, "\mathcal{E}^-_\alpha"] & \mathcal{E}_\alpha(EL)_- \end{tikzcd}$$

A discussion on the link between causality and the least action principle through this formalism is proposed in [**13**]. In particular, for trajectories in E^α_+, we may say that choosing the variations in E^α_- is not in contradiction with the causality principle, since they do not have a real physical meaning. From a physical point of view, while the trajectories may be qualified as "real", the variations remain only "virtual".

Now we apply the asymmetric fractional embedding on Lagrangian systems with more variables, in order to deal with equations such as (1.4) and (1.5).

5. Generalizations

5.1. Derivatives of higher orders. — Let $\alpha \in (0,1)$ and $k \geq 2$. We consider here generalized Lagrangian systems which involve derivatives up to order k.

Definition 13. — *An extended Lagrangian is a function*

$$\begin{array}{rccc} L: & \mathbb{R}^{n(k+1)} \times [a,b] & \longrightarrow & \mathbb{R} \\ & (y_0, y_1, \dots, y_k, t) & \longmapsto & L(y_0, y_1, \dots, y_k, t), \end{array}$$

which verifies the following properties:

– $L \in C^1(\mathbb{R}^{n(k+1)} \times [a,b])$,
– $\forall\, 1 \leq i \leq k,\ \partial_{i+1} L \in C^i(\mathbb{R}^{n(k+1)} \times [a,b])$.

The action is now defined by

$$\begin{array}{rccc} \mathcal{A}(L): & C^k([a,b]) & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \displaystyle\int_a^b L\left(x(t), \frac{d}{dt}x(t), \dots, \frac{d^k}{dt^k}x(t), t\right)\, dt. \end{array}$$

Similar results to Section 2.3 hold for those Lagrangians.

Lemma 10. — *Let L be an extended Lagrangian and $x \in C^k([a,b])$.*
We suppose that for all $1 \leq i \leq k$, $\partial_{i+1} L\left(x(\bullet), \frac{d}{dt}x(\bullet), \dots, \frac{d^k}{dt^k}x(\bullet), \bullet\right) \in AC^i([a,b])$.
Then $\mathcal{A}(L)$ is $C_0^k([a,b])$-differentiable at x and for all $h \in C_0^k([a,b])$,

$$d\mathcal{A}(L)(x,h) = \int_a^b \left[\partial_1 L + \sum_{i=1}^k (-1)^i \frac{d^i}{dt^i} \partial_{i+1} L\right] \left(x(t), \dots, \frac{d^k}{dt^k}x(t), t\right)\, dt.$$

Lemma 11. — *Let L be an extended Lagrangian and $x \in C^k([a,b])$.*
We suppose that for all $1 \leq i \leq k$, $\partial_{i+1} L\left(x(\bullet), \frac{d}{dt}x(\bullet), \dots, \frac{d^k}{dt^k}x(\bullet), \bullet\right) \in AC^i([a,b])$.
Then we have the following equivalence:
x is a $C_0^k([a,b])$-extremal for $\mathcal{A}(L)$ if and only if x verifies the Euler-Lagrange equation

$$(EL_k) \qquad \forall t \in [a,b], \quad \left[\partial_1 L + \sum_{i=1}^k (-1)^i \frac{d^i}{dt^i} \partial_{i+1} L\right] \left(x(t), \dots, \frac{d^k}{dt^k}x(t), t\right) = 0. \tag{5.1}$$

Concerning the asymmetric fractional embedding, we start with the embedding of the Euler-Lagrange equation.

The asymmetric fractional embedding of (5.1) is given by:

$$\mathcal{E}_\alpha(EL_k) \qquad \left[\partial_1 \tilde{L} + \sigma(X) \sum_{i=1}^k \begin{pmatrix} (-1)^i\, \mathcal{D}_+^{\alpha i}\, \partial_{i+1} \tilde{L} \\ \mathcal{D}_-^{\alpha i}\, \partial_{i+1} \tilde{L} \end{pmatrix}\right] \left(X(t), \dots, ({}^c\mathcal{D}^\alpha)^k X(t), t\right) = 0.$$

In particular, for $(x_+, 0)$, we have

$$\mathcal{E}_\alpha(EL_k)_+ \qquad \left[\partial_1 L + \sum_{i=1}^k (-1)^i\, \mathcal{D}_+^{\alpha i}\, \partial_{i+1} L\right] \left(x_+(t), \dots, ({}^c\mathcal{D}_+^\alpha)^k x_+(t), t\right) = 0, \tag{5.2}$$

and for $(0, x_-)$,

$$\mathcal{E}_\alpha(EL_k)_- \qquad \left[\partial_1 L + \sum_{i=1}^k \mathcal{D}_-^{\alpha i}\, \partial_{i+1} L\right] \left(x_-(t), \dots, (-\,{}^c\mathcal{D}_-^\alpha)^k x_-(t), t\right) = 0. \tag{5.3}$$

Now we consider the embedding of the extended Lagrangian. First we need to set a vector space for the trajectories, suitable for the calculus of variations. Let F_k^α be the functional space defined by

$$F^{\alpha,k}([a,b]) = \left\{ X \in C^0([a,b])^2 \mid \forall\, 1 \leq i \leq k,\ ({}^c\mathcal{D}^\alpha)^i X \in C^0([a,b])^2 \right\}.$$

We also introduce

$$F_+^{\alpha,k}([a,b]) = F^{\alpha,k}([a,b]) \cap \left(\mathcal{F}([a,b], \mathbb{R}^n) \times \{0\}\right),$$

$$F_{-}^{\alpha,k}([a,b]) = F^{\alpha,k}([a,b]) \cap (\{0\} \times \mathcal{F}([a,b],\mathbb{R}^n))\,.$$

The asymmetric fractional embedding of L, still denoted by L_α, is given by

$$\begin{aligned} L_\alpha(X)(t) &= \tilde{L}(X(t),\dots,({}^c\mathcal{D}^\alpha)^k X(t),t) \\ &= L(x_+(t)+x_-(t),\dots,({}^c\mathcal{D}^\alpha_+)^k x_+(t) + (-{}^c\mathcal{D}^\alpha_-)^k x_-(t),t), \end{aligned}$$

for all $X=(x_+,x_-) \in F^{\alpha,k}([a,b])$ and $t\in[a,b]$.

The associated action is now given by

$$\begin{array}{cccc} \mathcal{A}(L_\alpha): & F^{\alpha,k}([a,b]) & \longrightarrow & \mathbb{R} \\ & X & \longmapsto & \displaystyle\int_a^b \tilde{L}\Big(X(t),\, {}^c\mathcal{D}^\alpha X(t),\dots,({}^c\mathcal{D}^\alpha)^k X(t),t\Big)\,dt. \end{array}$$

The variations should be of course in $F^{\alpha,k}([a,b])$ and should be suitable for the integration by parts. The space $C^k_0([a,b])$ is suitable (but may not be optimal). In particular, $C^k_0([a,b]) \subset F^{\alpha,k}([a,b])$ from Lemmas 6 and 4.

The differential of the action is given by the following result.

Lemma 12. — *Let L be an extended Lagrangian and $X \in F^{\alpha,k}([a,b])$.*
We suppose that for all $1\le i\le k$, $\partial_{i+1}\tilde{L}(X(\bullet),\dots,({}^c\mathcal{D}^\alpha)^k X(\bullet),\bullet) \in AC^{\underline{\alpha i}}([a,b])$.
Then $\mathcal{A}(L_\alpha)$ is $C^k_0([a,b])^2$-differentiable at X and for all $H=(h_+,h_-)\in C^k_0([a,b])^2$,

$$\begin{aligned} d\mathcal{A}(L_\alpha)(X,H) = &\int_a^b \left[\partial_1\tilde{L} + \sum_{i=1}^k \mathcal{D}^{\alpha i}_-\,\partial_{i+1}\tilde{L}\right]\Big(X(t),\dots,({}^c\mathcal{D}^\alpha)^k X(t),t\Big)\cdot h_+(t)\,dt \\ &+ \int_a^b \left[\partial_1\tilde{L} + \sum_{i=1}^k (-1)^i\,\mathcal{D}^{\alpha i}_+\,\partial_{i+1}\tilde{L}\right]\Big(X(t),\dots,({}^c\mathcal{D}^\alpha)^k X(t),t\Big)\cdot h_-(t)\,dt. \end{aligned}$$

Proof. — Let $H=(h_+,h_-)\in C^k_0([a,b])^2$ and $\varepsilon>0$. Similarly to Lemma 9, we have:

$$\begin{aligned} \mathcal{A}(L_\alpha)(X+\varepsilon H) = \mathcal{A}(L_\alpha)(X) &+ \varepsilon\int_a^b \partial_1\tilde{L}(X(t),\dots,({}^c\mathcal{D}^\alpha)^k X(t),t)\cdot(h_+(t)+h_-(t))\,dt \\ &+ \varepsilon\int_a^b\sum_{i=1}^k \partial_{i+1}\tilde{L}(X(t),\dots,({}^c\mathcal{D}^\alpha)^k X(t),t)\cdot\Big(({}^c\mathcal{D}^\alpha_+)^i h_+(t) + (-{}^c\mathcal{D}^\alpha_-)^i h_-(t)\Big)\,dt + o(\varepsilon). \end{aligned}$$

Let $1\le i\le k$. Since $h_+\in C^i_0([a,b])$, it verifies $({}^c\mathcal{D}^\alpha_+)^i h_+ = {}^c\mathcal{D}^{\alpha i}_+$, from Lemma 6. By using Lemma 8, we have:

$$\int_a^b \partial_{i+1}\tilde{L}(\dots)\cdot {}^c\mathcal{D}^{\alpha i}_+\,h_+(t)\,dt = \int_a^b \mathcal{D}^{\alpha i}_-\,\partial_{i+1}\tilde{L}(\dots)\cdot h_+(t)\,dt.$$

A similar relation holds for h_-. Hence we obtain

$$\begin{aligned}\mathcal{A}(L_\alpha)(X+\varepsilon H) &= \mathcal{A}(L_\alpha)(X) \\ &\quad + \varepsilon \int_a^b \left[\partial_1 \tilde{L} + \sum_{i=1}^{k} \mathcal{D}_-^{\alpha i}\, \partial_{i+1}\tilde{L}\right]\Big(X(t),\dots,({}^c\mathcal{D}^\alpha)^k X(t),t\Big)\cdot h_+(t)\,dt \\ &\quad + \varepsilon \int_a^b \left[\partial_1 \tilde{L} + \sum_{i=1}^{k} (-1)^i\, \mathcal{D}_+^{\alpha i}\, \partial_{i+1}\tilde{L}\right]\Big(X(t),\dots,({}^c\mathcal{D}^\alpha)^k X(t),t\Big)\cdot h_-(t)\,dt + o(\varepsilon).\end{aligned}$$

The terms in h_+ and h_- are linear, which concludes the proof. □

We may still obtain coherent and causal embeddings, thanks to the following equivalences.

Theorem 5. — *Let $(x_+,0) \in F_+^{\alpha,k}([a,b])$.*
We suppose that for all $1 \le i \le k$, $\partial_{i+1}L(x_+(\bullet),\dots,({}^c\mathcal{D}_+^\alpha)^k x_+(\bullet),\bullet) \in AC^{\underline{\alpha i}}([a,b])$.
Then we have the following equivalence:
$(x_+,0)$ is a $\{0\}\times C_0^k([a,b])$-extremal of the action $\mathcal{A}(L_\alpha)$ if and only if x_+ verifies

$$(EL_{k,\alpha})_+ \qquad \forall t \in (a,b], \quad \left[\partial_1 L + \sum_{i=1}^{k} (-1)^i\, \mathcal{D}_+^{\alpha i}\, \partial_{i+1} L\right]\Big(x_+(t),\dots,({}^c\mathcal{D}_+^\alpha)^k x_+(t),t\Big) = 0. \tag{5.4}$$

Let $(0,x_-) \in F_-^{\alpha,k}([a,b])$.
We suppose that for all $1 \le i \le k$, $\partial_{i+1}L(x_-(\bullet),\dots,(-{}^c\mathcal{D}_-^\alpha)^k x_-(\bullet),\bullet) \in AC^{\overline{\alpha i}}([a,b])$.
Then we have the following equivalence:
$(0,x_-)$ is a $C_0^k([a,b])\times\{0\}$-extremal of the action $\mathcal{A}(L_\alpha)$ if and only if x_- verifies

$$(EL_{k,\alpha})_- \qquad \forall t \in [a,b), \quad \left[\partial_1 L + \sum_{i=1}^{k} \mathcal{D}_-^{\alpha i}\, \partial_{i+1} L\right]\Big(x_-(t),\dots,(-{}^c\mathcal{D}_-^\alpha)^k x_-(t),t\Big) = 0. \tag{5.5}$$

Proof. — From Theorem 1.2.4 of **[20]**, the fundamental lemma in the calculus of variations is still valid for variations in $C_0^\infty([a,b])$. Since $C_0^\infty([a,b]) \subset C_0^k([a,b])$, the result is proved. □

Equations (5.4) and (5.5) are once again similar to (5.2) and (5.3): $(EL_{k,\alpha})_\pm \equiv \mathcal{E}_\alpha(EL_k)_\pm$. The following diagrams are thus valid:

$$\begin{tikzcd} L \arrow[r, "\mathcal{E}_\alpha^+"] \arrow[d, leftrightarrow, "C_0^k({[a,b]})"'] & L_\alpha \arrow[d, leftrightarrow, "\{0\}\times C_0^k({[a,b]})"] \\ (EL_k) \arrow[r, "\mathcal{E}_\alpha^+"] & \mathcal{E}_\alpha(EL_k)_+ \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, "\mathcal{E}_\alpha^-"] \arrow[d, leftrightarrow, "C_0^k({[a,b]})"'] & L_\alpha \arrow[d, leftrightarrow, "C_0^k({[a,b]})\times\{0\}"] \\ (EL_k) \arrow[r, "\mathcal{E}_\alpha^-"] & \mathcal{E}_\alpha(EL_k)_- \end{tikzcd}$$

5.2. Continuous Lagrangian systems. — For the sake of simplicity, we do not generalize the fractional embedding for continuous Lagrangians, but the ideas would be the same. More details can be found in **[8]**. We only give a result which will be useful for the applications in the next section.

Let Ω be an open and regular subset of $\mathbb{R}^n$. We are no more interested by the evolution of a trajectory $x(t) \in \mathbb{R}^n$, but by the evolution of a field $u(x,t) \in \mathbb{R}$, with $(x,t) \in \Omega \times [a,b]$.

Hence we are interested by generalized Lagrangians of the form

$$\begin{array}{rccc} L: & \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R} & \longrightarrow & \mathbb{R} \\ & (u,v,w,x,t) & \longmapsto & L(u,v,w,x,t). \end{array}$$

In the classical case, the evaluation of this Lagrangian on a field $u(x,t)$ is

$$L(u(x,t),\nabla u(x,t),\partial_t u(x,t),x,t),$$

with $x\in\Omega$ and $t\in[a,b]$. The notation $\nabla u(x,t)\in\mathbb{R}^n$ is the gradient of $x\mapsto u(x,t)$ and $\partial_t u(x,t)\in\mathbb{R}$ the partial derivative of u according to t.

For such a Lagrangian, we define its asymmetric representation as

$$\begin{array}{rccc} \tilde{L}: & \mathbb{R}^2\times\mathbb{R}^{2n}\times\mathbb{R}^2\times\mathbb{R}^n\times\mathbb{R} & \longrightarrow & \mathbb{R} \\ & (u_1,u_2,v_1,v_2,w_1,w_2,x,t) & \longmapsto & L(u_1+u_2,v_1+v_2,w_1+w_2,x,t). \end{array}$$

We set $C_0^p(\Omega)=\{f\in C^p(\Omega)\mid f=0 \text{ on } \partial\Omega\}$ and

$$\begin{aligned} V^\alpha(\Omega\times[a,b]) = \big\{\ & h\in C^1(\Omega\times[a,b])\mid \forall t\in[a,b],\ x\mapsto h(x,t)\in C_0^0(\Omega), \\ & \forall x\in\Omega,\ h(x,a)=h(x,b)=0\ \big\}. \end{aligned}$$

Here the fractional derivatives are seen as partial fractional derivatives according to t. For instance, we note ${}^c\mathcal{D}^\alpha_+\,u(x,t) = {}^c\mathcal{D}^\alpha_+\,u_x(t)$, where $u_x : t\mapsto u(x,t)$. Similarly, for $U=(u_+,u_-)$, we note ${}^c\mathcal{D}^\alpha U(x,t) = ({}^c\mathcal{D}^\alpha_+\,u_+(x,t), -\,{}^c\mathcal{D}^\alpha_-\,u_-(x,t))$.

The action is now defined by

$$\begin{array}{rccc} \mathcal{A}(L_\alpha): & C^1(\Omega\times[a,b])^2 & \longrightarrow & \mathbb{R} \\ & U & \longmapsto & \displaystyle\int_a^b\int_\Omega \tilde{L}\left(U(x,t),\nabla U(x,t),\,{}^c\mathcal{D}^\alpha U(x,t),x,t\right)\,dx\,dt. \end{array}$$

For a generalized Lagrangian $L(u,v,w,x,t)$, we note in this section $\partial_u L$, $\partial_{v_i}L$ and ∂_w the partial derivatives of L according to its first, $i+1$-th and $n+2$-th variables.

Once again, we may obtain a causal Euler-Lagrange equation.

Theorem 6. — *Let $u_+\in C^1(\Omega\times[a,b])$.*
We suppose that

- $\forall\, x\in\Omega,\ t\mapsto \partial_w L(u_+(x,t),\nabla u_+(x,t),\,{}^c\mathcal{D}^\alpha_+\,u_+(x,t),x,t)\in AC([a,b])$,
- $\forall\, 1\le i\le n,\ \forall\, t\in[a,b],\ x\mapsto \partial_{v_i}L(u_+(x,t),\nabla u_+(x,t),\,{}^c\mathcal{D}^\alpha_+\,u_+(x,t),x,t)\in C^1(\Omega)$.

Then we have the following equivalence:
$(u_+,0)$ is a $\{0\}\times V^\alpha(\Omega\times[a,b])$-extremal of the action $\mathcal{A}(L_\alpha)$ if and only if u_+ verifies

$$\left[\partial_u L-\sum_{i=1}^n \partial_{x_i}\partial_{v_i}L-\ \mathcal{D}^\alpha_+\,\partial_w L\right](u_+(x,t),\nabla u_+(x,t),\,{}^c\mathcal{D}^\alpha_+\,u_+(x,t),x,t)=0,$$

for all $x\in\Omega$, $t\in(a,b]$.

Theorems 5 and 6 will now be used to associate variational formulations to equations (1.4) and (1.5).

6. Applications

6.1. Linear friction. — The differential equation of linear friction is

$$m\frac{d^2}{dt^2}x(t) + \gamma\frac{d}{dt}x(t) - \nabla U(x(t)) = 0, \tag{6.1}$$

where $t \in [a,b]$, $m, \gamma > 0$ and $U \in C^1(\mathbb{R}^n)$.

Even if $U(x)$ is quadratic, it has been shown in **[7]** that this equation cannot be derived from a variational principle with classical derivatives. But this can be done by using fractional derivatives, since $\dfrac{d}{dt} = {}^c\mathcal{D}_+^{1/2} \circ {}^c\mathcal{D}_+^{1/2}$, which is proved in the following lemma.

Lemma 13. — *If $f \in AC^2([a,b])$, we have:*
- *if $0 < \alpha < 1/2$, ${}^c\mathcal{D}_+^{\alpha} \circ {}^c\mathcal{D}_+^{\alpha} f = {}^c\mathcal{D}_+^{2\alpha} f$,*
- *if $\alpha = 1/2$, ${}^c\mathcal{D}_+^{1/2} \circ {}^c\mathcal{D}_+^{1/2} f = f'$,*
- *if $1/2 < \alpha < 1$, for all $t \in (a,b]$, ${}^c\mathcal{D}_+^{\alpha} \circ {}^c\mathcal{D}_+^{\alpha} f(t) = {}^c\mathcal{D}_+^{2\alpha} f(t) + \dfrac{f'(a)}{\Gamma(2-2\alpha)}(t-a)^{1-2\alpha}$.*

Proof. — See Section 8. □

We consider the function $L(x,v,w,t) = \dfrac{m}{2}w^2 - \dfrac{\gamma}{2}v^2 - U(x)$, which is an extended Lagrangian.

The variations should be chosen in $C_0^2([a,b])$, but the space

$$AC_0^2([a,b]) = \{f \in AC^2([a,b]) \mid f(a) = f(b) = 0\}$$

is actually sufficient.

Theorem 7. — *Let $x \in C^2([a,b])$. Then x is solution of* (6.1) *if and only if $(x,0)$ is a $\{0\} \times AC_0^2([a,b])$-extremal of the action $\mathcal{A}(L_{1/2})$.*

Proof. — From Lemma 13 and its proof, ${}^c\mathcal{D}_+^{1/2} \circ {}^c\mathcal{D}_+^{1/2} x(t) = \mathcal{D}_+^{1/2} \circ {}^c\mathcal{D}_+^{1/2} x(t) = \dfrac{d}{dt}x(t)$, for all $t \in [a,b]$. Hence $L(x(\bullet), {}^c\mathcal{D}_+^{1/2} x(\bullet), x'(\bullet), \bullet) \in C^0([a,b])$ and the action $\mathcal{A}(L_{1/2})$ is well defined.

Let $t \in [a,b]$. The partial derivatives of L verify:
- $\partial_1 L(x(t), {}^c\mathcal{D}_+^{1/2} x(t), x'(t), t) = -\nabla U(x(t))$,
- $\partial_2 L(x(t), {}^c\mathcal{D}_+^{1/2} x(t), x'(t), t) = -\gamma\, {}^c\mathcal{D}_+^{1/2} x(t)$,
- $\partial_3 L(x(t), {}^c\mathcal{D}_+^{1/2} x(t), x'(t), t) = m\, x'(t)$.

Since $x' \in AC([a,b])$, ${}^c\mathcal{D}_+^{1/2} x = \mathcal{I}_+^{1/2} x' \in AC([a,b])$, from Lemma 3. Consequently, for all $1 \le i \le 2$, $\partial_{i+1} L(x(\bullet), {}^c\mathcal{D}_+^{1/2} x(\bullet), x'(\bullet), \bullet) \in AC^{\underline{\alpha i}}([a,b])$. Conditions of application of Theorem 5 are hence fulfilled. Since $C_0^\infty([a,b]) \subset AC_0^2([a,b])$, the choice of $\{0\} \times AC_0^2([a,b])$ for the variations is valid and Theorem 5 may be applied:

$(x,0)$ is a $\{0\} \times AC_0^2([a,b])$-extremal of the action $\mathcal{A}(L_{1/2})$ if and only if x verifies

$$\left[\partial_1 L - \mathcal{D}_+^{1/2}\, \partial_2 L + \mathcal{D}_+^{1}\, \partial_3 L\right](x(t), {}^c\mathcal{D}_+^{1/2} x(t), x'(t), t) = 0. \tag{6.2}$$

Given that $\mathcal{D}_+^1\, x'(t) = x''(t)$, (6.2) is exactly (6.1). □

We see here the necessity of having a causal Euler-Lagrange equation. Indeed, an equation similar to (4.1) would have provide $\mathcal{D}^{\alpha}_{-} \circ {}^{c}\mathcal{D}^{\alpha}_{+}$ which is never equal to $\dfrac{d}{dt}$.

Furthermore, the choice of $({}^{c}\mathcal{D}^{\alpha})^k$ instead of ${}^{c}\mathcal{D}^{\alpha k}$ in the asymmetric fractional embedding is justified here. If we had taken ${}^{c}\mathcal{D}^{\alpha k}$, the evaluation of the Lagrangian in this example would have been $L(x(t), {}^{c}\mathcal{D}^{1/2}_{+} x(t), x'(t) - x'(a), t)$, since ${}^{c}\mathcal{D}^{1}_{+} x(t) = x'(t) - x'(a)$. Hence the initial condition $x'(a) = 0$ should have been added to obtain (6.1), which is too restrictive for the solutions of (6.1).

6.2. Diffusion equation. — We are now interested in the diffusion equation

$$\frac{\partial}{\partial t} u(x,t) = c\,\Delta u(x,t), \tag{6.3}$$

where $t \in [a,b]$, $x \in \Omega$, $c > 0$, and Δ is the Laplace operator.

We consider the generalized Lagrangian $L(u,v,w,x,t) = \dfrac{1}{2}w^2 - \dfrac{c}{2}v^2$.

Theorem 8. — *Let $u \in \mathcal{F}(\Omega \times [a,b], \mathbb{R})$ such that*

– $\forall x \in \Omega,\ t \mapsto u(x,t) \in AC^2([a,b])$,

– $\forall t \in [a,b],\ x \mapsto u(x,t) \in C^2(\Omega)$.

Then u is solution of (6.3) *if and only if $(u,0)$ is a $\{0\} \times V^{\alpha}(\Omega \times [a,b])$-extremal of the action $\mathcal{A}(L_{1/2})$.*

Proof. — Let $x \in \Omega$ and $t \in [a,b]$. The partial derivatives of L verify:

– $\partial_w L(u(x,t), \nabla u(x,t), {}^{c}\mathcal{D}^{1/2}_{+} u(x,t), x, t) = {}^{c}\mathcal{D}^{1/2}_{+} u(x,t)$,

– $\forall 1 \leq i \leq n,\ \partial_{v_i} L(u(x,t), \nabla u(x,t), {}^{c}\mathcal{D}^{1/2}_{+} u(x,t), x, t) = -c\,\partial_{x_i} u(x,t)$,

so conditions of Theorem 6 are fulfilled, and we have:

$(u,0)$ is a $\{0\} \times V^{\alpha}(\Omega \times [a,b])$-extremal of the action $\mathcal{A}(L_{1/2})$ if and only if u verifies

$$\left[\partial_u L - \sum_{i=1}^{n} \partial_{x_i}\partial_{v_i} L - \mathcal{D}^{1/2}_{+}\,\partial_w L\right](u(x,t), \nabla u(x,t), {}^{c}\mathcal{D}^{1/2}_{+} u(x,t), x, t) = 0. \tag{6.4}$$

Given that $\displaystyle\sum_{i=1}^{n} \partial_{x_i}\partial_{x_i} u(x,t) = \Delta u(x,t)$ and $\mathcal{D}^{1/2}_{+} \circ {}^{c}\mathcal{D}^{1/2}_{+} u(x,t) = \dfrac{\partial}{\partial t} u(x,t)$, (6.4) is exactly (6.3).

□

Once again, causality is essential so as to obtain the term $\dfrac{\partial}{\partial t} u(x,t)$.

7. Conclusion

In this article we have proposed an asymmetric fractional embedding, which turns out to be coherent and causal. Such an embedding provides a strong variational structure for friction and diffusion equations, in the sense that solutions of those two equations are exactly the extremal of some functionals.

Furthermore, a rigorous treatment of the fractional operators has been carried out all along the article, even if some classes of functions may not be optimal. In particular, it has been suggested that the Caputo derivative could be preferred to the Riemann-Liouville derivative for the definition of the fractional Lagrangian action.

Concerning the dimensional homogeneity of fractional derivatives and equations, the asymmetric fractional embedding is compatible with the homogeneous fractional embedding presented in [**21**]. For instance, if τ is the extrinsic constant of time, the use of $\tau^{\alpha-1}\mathcal{D}^\alpha$ instead of $\mathcal{D}^\alpha$ for all the four fractional derivatives provides fractional differential equations which are homogeneous in time.

We have the following list of open problems and perspectives :

- One must develop the critical point theory associated to our fractional functionals in order to provide results about existence and regularity of solutions for these PDEs.

- Our paper solve the inverse problem of the fractional calculus of variations for some classical or fractional PDEs (classical or fractional diffusion equation, fractional wave equation, convection-diffusion (see [**11**]). However, we have no characterization of PDEs admitting a fractional variational formulation in our setting. In the classical case, the Lie approach to ODEs or PDEs as exposed for example in [**27**] provides a necessary criteria known as Helmholtz's conditions (see [**27**],p.). A natural idea is to look for the corresponding theory in our case.

- There exists suitable numerical algorithms to study classical Lagrangian systems called *variational integrators* which are developed for example in [**23**], [**24**]. The basic idea of a variational integrator is to preserve this variational structure at the discrete level. A natural extension of our work is then to develop variational integrators adapted to our fractional Lagrangian functionals. A first step in this direction has been done in [**12**] by introducing the notion of discrete embedding of Lagrangian systems. However, this work does not cover continuous fractional Lagrangian systems and uses only classical discretization of the Riemann-Liouville or Caputo derivative by classical Grünwald-Leitnikov expansions. However for classical functionals we have extended this point of view to finite-elements and finite-volumes methods [**?**]. We will discuss the case of continuous fractional Lagrangian systems in a forthcoming paper.

Of course of all these problems are far from being solved for the moment. However, it proves that fractional calculus can be useful in a number of classical problems of Analysis and in particular for PDEs where classical methods do not provide efficient tools.

8. Proofs

For $x > 0$ the integers $\overline{x}$ and $\underline{x}$ are defined by $\overline{x} - 1 \leq x < \overline{x}$ and $\underline{x} - 1 < x \leq \underline{x}$. A preliminary lemma will be useful for the following ones.

Lemma 14. — *Let $\beta > 0$ and $p \in \mathbb{N}^*$. If $f \in C^p_+([a,b])$, then* $\dfrac{d^p}{dt^p} \circ \mathcal{I}^\beta_+ f = \mathcal{I}^\beta_+ \circ \dfrac{d^p}{dt^p} f$.

Proof. — We prove it by induction. For $p = 1$ and $f \in C^1_+([a,b])$,

$$\begin{aligned}\frac{d}{dt}\,\mathcal{I}^{\beta}_{+}\,f(t) &= \frac{(t-a)^{\beta-1}}{\Gamma(\beta)}f(a) + \frac{1}{\Gamma(\beta)}\int_a^t (t-\tau)^{\beta-1} f'(\tau)\,d\tau \\ &= \mathcal{I}^{\beta}_{+}\,f'(t),\end{aligned}$$

since $f(a) = 0$.

Now let $p \in \mathbb{N}^*$ and $f \in C^{p+1}_+([a,b])$. Since $f' \in C^p_+([a,b])$, we may apply the induction hypothesis:

$$\frac{d^p}{dt^p} \circ \mathcal{I}^{\beta}_{+}\,f' = \mathcal{I}^{\beta}_{+} \circ \frac{d^p}{dt^p} f'. \tag{8.1}$$

From case $p = 1$, $\mathcal{I}^{\beta}_{+}\,f' = \dfrac{d}{dt}\mathcal{I}^{\beta}_{+}\,f$. Hence (8.1) becomes

$$\frac{d^{p+1}}{dt^{p+1}} \circ \mathcal{I}^{\beta}_{+}\,f = \mathcal{I}^{\beta}_{+} \circ \frac{d^{p+1}}{dt^{p+1}} f,$$

which concludes the proof. □

8.1. Lemma 5. —

Proof. — Let $X = (x_+, x_-) \in C^1([a,b])^2$. Since $x'_+ \in C^0([a,b])$, all points of (a,b) are Lebesgue points of x'. We may then apply Theorem 2.7 of [**32**, p.51]:

$$\forall\, t \in (a,b),\ \lim_{\alpha \to 1^-} \mathcal{I}^{1-\alpha}_{+}\,x'(t) = x'(t).$$

We proceed likewise for x_-. □

8.2. Lemma 6. —

Proof. — We prove it by induction on k. For $k = 1$, the result is obvious. Now, let $k \in \mathbb{N}^*$ and $f \in C^{k+1}_+([a,b])$. Since $f \in C^k_+([a,b])$, we use the induction hypothesis: $({}^c\mathcal{D}^{\alpha}_{+})^k f = {}^c\mathcal{D}^{\alpha k}_{+}\,f = \mathcal{I}^{\overline{\alpha k}-\alpha k}_{+}\,f^{(\overline{\alpha k})}$. We have $\overline{\alpha k} \leq k$, so $f^{(\overline{\alpha k})} \in AC([a,b])$, and from Lemma 3, $({}^c\mathcal{D}^{\alpha}_{+})^k f \in AC([a,b])$. Moreover, from Lemma 4, $({}^c\mathcal{D}^{\alpha}_{+})^k f(a) = 0$. We may then apply Theorem 2:

$$\mathcal{D}^{\alpha}_{+} \circ ({}^c\mathcal{D}^{\alpha}_{+})^k f = {}^c\mathcal{D}^{\alpha}_{+} \circ ({}^c\mathcal{D}^{\alpha}_{+})^k f = ({}^c\mathcal{D}^{\alpha}_{+})^{k+1} f.$$

On the other hand, $\mathcal{D}^{\alpha}_{+} \circ ({}^c\mathcal{D}^{\alpha}_{+})^k f = \dfrac{d}{dt} \circ \mathcal{I}^{1-\alpha}_{+} \circ \mathcal{I}^{\overline{\alpha k}-\alpha k}_{+}\,f^{(\overline{\alpha k})}$. We have $f^{(\overline{\alpha k})} \in C^0([a,b])$, so we may use formula 2.21 of [**32**, p.34]:

$$\mathcal{I}^{1-\alpha}_{+} \circ \mathcal{I}^{\overline{\alpha k}-\alpha k}_{+}\,f^{(\overline{\alpha k})} = \mathcal{I}^{\beta}_{+}\,f^{(\overline{\alpha k})},$$

where $\beta = 1 + \overline{\alpha k} - \alpha(k+1)$.

Since $f^{(\overline{\alpha k})} \in C^1_+([a,b])$, from Lemma 14,

$$\frac{d}{dt} \circ \mathcal{I}^{\beta}_{+}\,f^{(\overline{\alpha k})} = \mathcal{I}^{\beta}_{+}\,f^{(\overline{\alpha k}+1)}.$$

We have $\overline{\alpha k} + 1 \in \{\overline{\alpha(k+1)},\ \overline{\alpha(k+1)} + 1\}$, so we consider two cases.

– If $\overline{\alpha k} + 1 = \overline{\alpha(k+1)}$, then

$$\begin{aligned}\mathcal{D}^{\alpha}_{+} \circ ({}^c\mathcal{D}^{\alpha}_{+})^k f &= \mathcal{I}^{\overline{\alpha(k+1)}-\alpha(k+1)}_{+}\,f^{(\overline{\alpha(k+1)})}, \\ &= {}^c\mathcal{D}^{\alpha(k+1)}_{+}\,f.\end{aligned}$$

– If $\overline{\alpha k} = \overline{\alpha(k+1)}$, then

$$\begin{aligned}\mathcal{D}_{+}^{\alpha} \circ ({}^{c}\mathcal{D}_{+}^{\alpha})^{k} f &= \mathcal{I}_{+}^{1+\overline{\alpha(k+1)}-\alpha(k+1)} f^{(\overline{\alpha(k+1)}+1)},\\ &= \mathcal{I}_{+}^{\overline{\alpha(k+1)}-\alpha(k+1)} \circ \mathcal{I}_{+}^{1} f^{(\overline{\alpha(k+1)}+1)}.\end{aligned}$$

We have $\mathcal{I}_{+}^{1} f^{(\overline{\alpha(k+1)}+1)}(t) = f^{(\overline{\alpha(k+1)})}(t) - f^{(\overline{\alpha(k+1)})}(a)$. But $\overline{\alpha(k+1)} \leq k$, so $f^{(\overline{\alpha(k+1)})}(a) = 0$.

Consequently,

$$\begin{aligned}\mathcal{D}_{+}^{\alpha} \circ ({}^{c}\mathcal{D}_{+}^{\alpha})^{k} f &= \mathcal{I}_{+}^{\overline{\alpha(k+1)}-\alpha(k+1)} f^{(\overline{\alpha(k+1)})},\\ &= {}^{c}\mathcal{D}_{+}^{\alpha(k+1)} f.\end{aligned}$$

In both cases, we have proved that $({}^{c}\mathcal{D}_{+}^{\alpha})^{k+1} f = {}^{c}\mathcal{D}_{+}^{\alpha(k+1)} f$, which concludes the proof. □

8.3. Lemma 7. —

Proof. — If $\beta \in \mathbb{N}^*$, ${}^{c}\mathcal{D}_{+}^{\beta} f(t) = f^{(\beta)}(t) - f^{(\beta)}(a)$, and ${}^{c}\mathcal{D}_{+}^{\beta} f \in C_{+}^{p}([a,b])$. Else, let $1 \leq k \leq p$. Since $f^{(\underline{\beta})} \in C_{+}^{k}([a,b])$, from Lemma 14,

$$\begin{aligned}\frac{d^k}{dt^k}\, {}^{c}\mathcal{D}_{+}^{\beta} f &= \frac{d^k}{dt^k} \mathcal{I}_{+}^{\underline{\beta}-\beta} f^{(\underline{\beta})}\\ &= \mathcal{I}_{+}^{\underline{\beta}-\beta} f^{(\underline{\beta}+k)}\end{aligned}$$

Given that $f^{(\underline{\beta}+k)} \in C^0([a,b])$, $\mathcal{I}_{+}^{\underline{\beta}-\beta} f^{(\underline{\beta}+k)} \in C_{+}^{0}([a,b])$, from Lemma 3. Hence ${}^{c}\mathcal{D}_{+}^{\beta} f \in C^k([a,b])$ and $\dfrac{d^k}{dt^k}\, {}^{c}\mathcal{D}_{+}^{\beta} f(a) = 0$. Moreover, ${}^{c}\mathcal{D}_{+}^{\beta} f(a) = 0$ from Lemma 4. Finally, ${}^{c}\mathcal{D}_{+}^{\beta} f \in C_{+}^{p}([a,b])$. □

8.4. Lemma 8. —

Proof. — If $\beta \in \mathbb{N}^*$, this is the classical formula for integration by parts. Else, since $g \in C_0^{\underline{\beta}}([a,b])$, $g^{\underline{\beta}} \in L_p([a,b])$, with $p \geq 1/\beta$. Furthermore, $f^{(\underline{\beta})} \in L_1([a,b])$, so equation 2.20 of [**32**, p.34] is valid:

$$\int_a^b f(t) \cdot {}^{c}\mathcal{D}_{-}^{\beta} g(t)\, dt = (-1)^{\underline{\beta}} \int_a^b \mathcal{I}_{+}^{\underline{\beta}-\beta} f(t) \cdot g^{(\underline{\beta})}(t)\, dt.$$

Moreover, for all $0 \leq k \leq \underline{\beta} - 1$, $g^{(k)}(a) = g^{(k)}(b) = 0$. Therefore, iterating the classical integration by parts $\underline{\beta}$ times leads to:

$$(-1)^{\underline{\beta}} \int_a^b \mathcal{I}_{+}^{\underline{\beta}-\beta} f(t) \cdot g^{(\underline{\beta})}(t)\, dt = \int_a^b \frac{d^{\underline{\beta}}}{dt^{\underline{\beta}}} \mathcal{I}_{+}^{\underline{\beta}-\beta} f(t) \cdot g(t)\, dt = \int_a^b \mathcal{D}_{+}^{\beta} f(t) \cdot g(t)\, dt.$$

We proceed likewise for the other relation. □

8.5. Lemma 13. —

Proof. — The beginning of the proof of Lemma 6 with $k = 1$ is valid with the hypothesis $f \in AC^2([a,b])$: it shows that ${}^c\mathcal{D}_+^\alpha f \in AC([a,b])$ and ${}^c\mathcal{D}_+^\alpha \circ {}^c\mathcal{D}_+^\alpha f = \frac{d}{dt}\mathcal{I}_+^{2-2\alpha} f'$.

– If $0 < \alpha < 1/2$, for all $t \in [a,b]$,

$$\begin{aligned} \frac{d}{dt}\mathcal{I}_+^{2-2\alpha} f'(t) &= \frac{d}{dt}\mathcal{I}_+^{1-2\alpha}(f(t) - f(a)) \\ &= \mathcal{D}_+^{2\alpha} h(t), \end{aligned}$$

with $h(t) = f(t) - f(a)$. Since $h \in C_+^1([a,b])$, $\mathcal{D}_+^{2\alpha} h = {}^c\mathcal{D}_+^{2\alpha} h$. Finally ${}^c\mathcal{D}_+^{2\alpha} h = {}^c\mathcal{D}_+^{2\alpha} f$.

– If $\alpha = 1/2$, for all $t \in [a,b]$, $\frac{d}{dt}\mathcal{I}_+^{2-2\alpha} f'(t) = \frac{d}{dt}[f(t) - f(a)] = f'(t)$.

– If $1/2 < \alpha < 1$, for all $t \in (a,b]$,

$$\begin{aligned} \frac{d}{dt}\mathcal{I}_+^{2-2\alpha} f'(t) &= \mathcal{D}_+^{2\alpha-1} f'(t) \\ &= {}^c\mathcal{D}_+^{2\alpha-1} f'(t) + \frac{f'(a)}{\Gamma(2-2\alpha)}(t-a)^{1-2\alpha}, \end{aligned}$$

from Theorem 2 applied to $f' \in AC([a,b])$. Finally,

$$ {}^c\mathcal{D}_+^\alpha \circ {}^c\mathcal{D}_+^\alpha f(t) = {}^c\mathcal{D}_+^{2\alpha} f(t) + \frac{f'(a)}{\Gamma(2-2\alpha)}(t-a)^{1-2\alpha}. $$

□

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