HYPERBOLICITY VERSUS PARTIAL-HYPERBOLICITY AND THE TRANSVERSALITY-TORSION PHENOMENON

by

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Abstract. — In this paper, we describe a process to create hyperbolicity in the neighbourhood of a homoclinic orbit to a partially hyperbolic torus for three degrees of freedom Hamiltonian systems: the transversality-torsion phenomenon.

keywords: Hyperbolicity, Partially hyperbolic tori, Hamiltonian systems.

1. Introduction

The aim of this paper is to describe a process of creation of hyperbolicity in a *partially hyperbolic* context called the *transversality-torsion phenomenon* introduced in ([**4**],[**5**]). This process comes from the study of instability (Arnold diffusion) for (at least) three degrees of freedom near-integrable Hamiltonian systems [**1**] and more precisely from the derivation of a Smale-Birkhoff theorem ([**6**],[**5**]) for transversal homoclinic partially hyperbolic tori which come along multiple resonances [**14**]. Our starting point is the following conjecture of R.W. Easton ([**6**],p.252) about *symbolic dynamics* for transversal homoclinic partially hyperbolic tori: In [**6**], Easton has proved the existence of symbolic dynamics in a neighbourhood of a partially hyperbolic torus whose stable and unstable manifolds intersect transversally (in a given energy manifold containing the torus). This result is obtained under a stringent assumption on the linear part of the *homoclinic map* (see §.3 for a definition and [**6**],p.244), called the *homoclinic matrix*. However, Easton has conjectured ([**6**],p.252) that this assumption can be weakened, or perhaps cancelled.

In ([**4**],[**5**]), we have weakened the homoclinic matrix condition, but mainly, we have put in evidence a dynamical and geometrical phenomenon at the origin of the *hyperbolic nature* of symbolic dynamics, that we have called the *transversality-torsion phenomenon*: the transversality of the stable and unstable manifold of the torus coupled with the torsion of the flow around the torus give rise to a hyperbolic dynamics in the neighbourhood of the homoclinic connection. Our terminology is now commonly used and the transversality-torsion phenomenon has been studied and extended. We refer in particular to the papers of M. Gidea and C. Robinson ([**9**] p.64) and M. Gidea and R. De La LLave [**8**], dealing with topological methods in dynamics.

In this paper, we prove that the transversality-torsion phenomenon observed in a particular case in [**5**] arises in a generic situation for three degrees of freedom Hamiltonian systems.

The plan of the paper is the following: In §.2, we define transversal homoclinic partially hyperbolic tori. In §.3, we state the *hyperbolicity problem*, which can be resumed as finding the minimal conditions (about the dynamics on the torus and the geometry of the intersection of the stable and unstable manifolds) in order to have a *homoclinic transition map*[(1)] hyperbolic. In §.4, we solve the hyperbolicity problem for three degrees of freedom Hamiltonian systems, putting in evidence the transversality-torsion phenomenon, i.e. the fundamental role of the torsion of the flow on the torus and the transversality of the stable and unstable manifolds to induce hyperbolicity of the transition map.

2. Transversal homoclinic partially hyperbolic tori

In this section, we define partially hyperbolic tori following the paper of S. Bolotin and D. Treschev [**3**].

2.1. Partially hyperbolic tori. — Let $\mathcal{M}$ be a $2m$ dimensional symplectic manifold, and H an analytic Hamiltonian defined on $\mathcal{M}$.

Definition 1. — *A weakly reducible, diophantine partially hyperbolic torus for H is a torus for which there exists an analytic symplectic coordinates system, such that the Hamiltonian takes the form*

$$H(\theta, I, s, u) = \omega.I + \frac{1}{2}AI.I + s.M(\theta)u + O_3(I, s, u), \tag{1}$$

where $(\theta, I, s, u) \in \mathbb{T}^k \times \mathbb{R}^k \times \mathbb{R}^{m-k} \times \mathbb{R}^{m-k}$, with the symplectic structure $\nu = dI \wedge d\theta + ds \wedge du$, A is a $k \times k$ symmetric constant matrix, M is a definite positive matrix and for all $k \in \mathbb{Z}^n \setminus \{0\}$, we have

$$\mid \omega.k \mid \geq \alpha \mid k \mid^{-\beta}, \ \alpha, \beta > 0.$$

If M is a constant matrix, then the partially hyperbolic torus is say to be reducible.

(1) See §.3.1 for a definition.

In ([**3**], Theorem 1, p. 406), S. Bolotin and D. Treschev prove that this "KAM" definition is equivalent to the dynamical one (see [**3**], Definition 1 and 3, p. 402). Moreover, for $k = 1$ and $k = m - 1$, the torus is always reducible.

In [**3**], Bolotin and Treschev introduce the notion of *nondegenerate* hyperbolic torus, which is a condition of dynamical nature (see [**3**], definition 3, p.402). In the setting of weakly reducible hyperbolic tori, we can use the following definition which is equivalent to the dynamical one (see [**3**], Proposition 2,p. 404):

Definition 2. — *A weakly reducible hyperbolic torus is nondegenerate if* $\det A \neq 0$.

H. Eliasson [**7**] and L. Niederman [**13**], have proved the following normal form theorem for $m - 1$ dimensional tori:

Theorem 1. — *Let T be a $m - 1$ dimensional reducible and non-degenerate diophantine partially hyperbolic torus. There exists an analytic coordinates system (x, y, z^+, z^-) defined in a neighbourhood V of T, such that*

$$H = \omega.y + \lambda z^- z^+ + O_2(y, z^+ z^-). \tag{2}$$

The geometry of the torus can then be easily described ([**3**]): it admits analytic stable (resp. unstable) manifold, denoted by $W^+(T)$ (resp. $W^-(T)$), and locally defined in V by:

$$\begin{array}{rcl} W^+(T) & = & \{(x, y, z^+, z^-) \in V,\ y = 0, z^- = 0\}, \\ W^-(T) & = & \{(x, y, z^+, z^-) \in V,\ y = 0, z^+ = 0\}. \end{array} \tag{3}$$

2.2. Transversal homoclinic connection. — In the following, we denote by $\mathcal{H}$ the energy submanifold of $\mathcal{M}$ containing the torus under consideration. For convenience, a weakly reducible diophantine partially hyperbolic torus will be called a partially hyperbolic torus.

Definition 3. — *Let T be a $m-1$ dimensional partially hyperbolic torus. We say that T possesses a transversal homoclinic connection if its stable and unstable manifolds intersect transversally in $\mathcal{H}$.*

In this paper, we explore the existence of a *hyperbolic* dynamics in a neighbourhood of a transversal homoclinic connection to a partially hyperbolic torus.

3. The hyperbolicity problem

3.1. Set-up. — Let H be a m degree of freedom Hamiltonian system. Let T be a $m-1$ dimensional partially hyperbolic torus of H possessing a transversal homoclinic connection along a homoclinic (at least one) orbit denoted by γ. We introduce the following notations and terminology:

Let V be the Eliasson's normal form domain (2). There exists ([**11**]), a Poincaré section $\mathcal{S}$ of T in V, and an analytic coordinates systems in $\mathcal{S}$, denoted by $(\phi, \rho, s, u) \in \mathbb{T}^{m-2} \times \mathbb{R} \times \mathbb{R}^{m-2} \times \mathbb{R}$, such that the Poincaré map takes the form

$$f(\phi, s, \rho, u) = (\phi + \omega + \nu\rho, \lambda s, \rho, \lambda^{-1}u) + O_2(\rho, s, u), \tag{4}$$

where $\omega \in \mathbb{R}^{m-2}$, $\nu \in \mathbb{R}^{m-2}$, $0 < \lambda < 1$, $\nu\rho = (\nu_1\rho_1, \dots, \nu_{m-2}\rho_{m-2})$.

We denote by $f_l(\phi, s, \rho, u) = (\phi + \omega + \nu\rho, \lambda s, \rho, \lambda^{-1}u)$ the linear part of f.

We say that the torus T is with *torsion* if $\nu_i \neq 0$, for $i = 1, \dots, m-2$, and *without torsion* otherwise. We note that a torus is with torsion if and only if it is nondegenerate.

Let $p^- = (\phi^-, 0, 0, u^-) \in \mathcal{S}$ and $p^+ = (\phi^+, s^+, 0, 0) \in \mathcal{S}$, be the last (resp. the first) point of intersection between γ and $\mathcal{S}$ along the unstable manifold (resp. the stable manifold). There exists neighbourhoods V^+ and V^- in $\mathcal{S}$ of p^+ and p^- respectively, and a map $\Gamma : V^- \to V^+$, called the *homoclinic map*, such that $\Gamma(p^-) = p^+$. The homoclinic map is of the form

$$\Gamma(p^- + z) = p^+ + \Pi.z + O_2(z),$$

where Π is a matrix, called the *homoclinic matrix.* We denote by

$$\Gamma_l(p^- + z) = p^+ + \Pi.z.$$

We denote by $D_n = \{z \in V^+ \mid f_l^n(z) \in V^-\}$ and $D = \bigcup_{n\geq 1} D_n$. We denote by $\psi : D \to V^-$, the *transverse map* introduced by Jürgen Moser [**12**] and defined by

$$\psi(z) = f^n(z) \text{ if } z \in D_n.$$

We denote by $\psi_l(z) = f_l^n(z)$ if $z \in D_n$.

The differential of f_l, denoted by Df_l is the matrix

$$Df_l = \begin{pmatrix} \mathrm{Id} & 0 & \mathcal{V} & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}, \tag{5}$$

where Id is the $(m-2) \times (m-2)$ identity matrix and $\mathcal{V}$ the diagonal matrix with components ν_i, $i = 1, \dots, m-2$.

In the following, we always work in the Poincaré section $\mathcal{S}$.

Let $C = \{(u,v) \in \mathbb{R}^{m-1} \times \mathbb{R}^{m-1} \mid \parallel u \parallel_1 \leq 1, \parallel v \parallel_1 \leq 1\}$. We denote by $W_\mu : C \to V^+$, what we call an *Easton's window* (or simply window in the following) defined by

$$W_\mu(z) = \mu z + p^+.$$

We consider the map $\Delta : C \to C$, defined by

$$\Delta = (W_\mu)^{-1} \circ \Gamma \circ \psi \circ W_\mu.$$

We denote by

$$\Delta_l = (W_\mu)^{-1} \circ \Gamma_l \circ \psi_l \circ W_\mu.$$

The map $\Gamma \circ \psi$ is called the *homoclinic transition map.*

In the following, we call *linear model* a Hamiltonian system possessing a transversal homoclinic partially hyperbolic torus T such that the preceding maps are linear in a given coordinates systems.

3.2. The hyperbolicity problem. — We keep the notations and terminology of the previous section. For all matrix M, we denote by $\mathrm{spec}(M)$ its spectrum. The *hyperbolicity problem* can be formulated as follow:

Hyperbolicity Problem – *Let H be a m degrees of freedom Hamiltonian system. Let T be a $m-1$ dimensional partially hyperbolic of H possessing a transversal homoclinic connection. Under which conditions on n, ν and Π do we have*

$$\mathrm{spec}(\Pi.Df_l^n) \cap S^1 = \emptyset,$$

where $S^1 = \{z \in \mathbb{C}, \mid z \mid = 1\}$ is the unit circle in $\mathbb{C}$.

This problem is difficult as there exists no results about localization of eigenvalues for the product of two matrices[(2)]. In §.4, we solve the hyperbolicity problem in the three degrees of freedom case. We also prove (see §.4.3) that if $\mathrm{spec}(\Pi.Df_l^n) \cap S^1 = \emptyset$, then for μ sufficiently small and under additional assumptions on the remainders of f and Γ, the homoclinic map Δ is hyperbolic in a given neighbourhood of the homoclinic orbit.

[(2)]There exists hyperbolicity results for random or deterministic product of matrices like [**2**]. However, they are based on genericity arguments which can not be used in order to understand the role of each of the elements n, ν and Π in the creation of hyperbolicity.

4. The transversality-torsion phenomenon

In this section, we deal with three degrees of freedom Hamiltonian systems. In the following, we denote by $\mathcal{M}_{n,p}(\mathbb{R})$ the set of $n \times p$ matrices with real coefficients and for all matrices $M \in \mathcal{M}_{n\times n}(\mathbb{R})$, we denote by $\mid M \mid$ its determinant.

4.1. Transversality constraints. — The matrix $\Pi \in \mathcal{M}_{4,4}(\mathbb{R})$ has the following form in the symplectic base $(e_\phi, e_s, e_\rho, e_u)$:

$$\Pi = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A, B, C, D \in \mathcal{M}_{2,2}(\mathbb{R})$.

For all differentiable manifold $\mathcal{M}$, we denote by $T_x\mathcal{M}$ the tangent space to $\mathcal{M}$ at point $x \in \mathcal{M}$.

Definition 4. — *We say that the homoclinic matrix is transverse if and only if it satisfies the following transversality condition* $\Pi(T_{p^-}W^-(T)) + T_{p^+}W^+(T) = T_{p^+}\mathcal{S}$.

Of course, if the intersection of the stable and unstable manifold, $W^+(T)$ and $W^-(T)$, of a torus T is transverse along an homoclinic orbit γ, then the homoclinic matrix satisfies the transversality conditions by definition.

Lemma 1. — *The matrix* Π *is transverse if and only if* $\Delta = \begin{vmatrix} c_{1,1} & d_{1,2} \\ c_{2,1} & d_{2,2} \end{vmatrix} \neq 0$.

Proof. — Let $v = (v_\phi, 0, 0, v_u)$ be a vector in $T_{p^-}W^-(T)$. We have

$$\Pi v = (a_{11}v_\phi + b_{12}v_u, a_{21}v_\phi + b_{22}v_u, c_{11}v_\phi + d_{12}v_u, c_{21}v_\phi + d_{22}v_u). \tag{6}$$

We begins with the global condition of transversality, namely that $v' = \Pi v = (v'_\phi, v'_s, v'_\rho, v'_u)$ is such that $v'_\rho = 0$ and $v'_u = 0$ if and only if $v_\phi = 0$ and $v_u = 0$. This condition implies

$$\begin{vmatrix} c_{1,1} & d_{1,2} \\ c_{2,1} & d_{2,2} \end{vmatrix} \neq 0.$$

□

In the following, we need the following strengthening of the transversality condition:

Definition 5. — *The matrix* Π *is strongly transverse if* $\Delta \neq 0$ *and* $d_{2,2} \neq 0$.

The condition $d_{2,2} \neq 0$ does not come from the transversality assumption. We can understand the geometrical nature of this condition as follow:

The unstable (resp. stable) manifold $W^u(T)$ (resp. $W^s(T)$) is foliated by 1 dimensional manifolds (see [**15**],p.138) denoted by $W^u_p(T)$ (resp. $W^s_p(T)$), $p \in T$ (the Fenichel fibers), and

$$W^u(T) = \bigcup_{p\in T} W^u_p(T) \ \ (\text{resp. } W^s(T) = \bigcup_{p\in T} W^s_p(T)\,). \tag{7}$$

In the normal form coordinates system, we have for all $p = (\phi_p, 0, 0, 0) \in T$,

$$W^u_{(\phi_p,0,0,0)}(T) = \{(\phi, s, \rho, u) \in \mathbb{T}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \mid \phi = \phi_p,\ s = 0,\ \rho = 0\}, \tag{8}$$

$$W^s_{(\phi_p,0,0,0)}(T) = \{(\phi, s, \rho, u) \in \mathbb{T}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \mid \phi = \phi_p,\ u = 0,\ \rho = 0\}. \tag{9}$$

The condition $d_{2,2} \neq 0$ is then equivalent to the following geometrical condition on the foliation of the stable and unstable manifolds in the linear model.

Lemma 2. — *Let us consider the linear model. The condition $d_{2,2} \neq 0$ is equivalent to the transversality of the intersection between the unstable leave at $(\phi_-, 0, 0, 0)$ denoted by $W^u_{(\phi_-,0,0,0)}(T)$ with the invariant manifold defined by $\{(\phi, s, \rho, u) \in \mathbb{T}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R};\ u = 0\}$ at point $(\phi_+, s_+, 0, 0)$.*

4.2. The transversality-torsion phenomenon. — The main technical result of this paper is the following:

***Theorem 2* (Transversality-torsion phenomenon)**. — *Let H be a 3 degrees of freedom Hamiltonian system possessing a 2 dimensional partially hyperbolic tori with a transversal homoclinic connection. We keep notations from section 3. We assume that:*

- *i) The homoclinic matrix Π is transverse;*
- *ii) The torus is with torsion;*

Then, for n sufficiently large, the matrix $\Pi D f_l^n$ is hyperbolic.

Moreover, if the matrix Π is strongly transverse, i.e. $d_{22} \neq 0$, all its eigenvalues are reals and given asymptotically by

$$x_1(n) \sim -n\nu d_{22}^{-1}\Delta,\ x_2(n) \sim d_{22}\lambda^{-n},\ x_3(n) = x_1(n)^{-1},\ x_4(n) = x_2(n)^{-1},$$

where λ and ν are associated to Df_l of the form (5)

Proof. — Let us assume that the matrix $\Pi D f_l^n$ possesses a complex eigenvalue β_n. As $\Pi D f_l^n$ is symplectic, we know that the three remanning eigenvalues are $\bar{\beta_n}$, $1/\beta_n$ and $1/\bar{\beta_n}$ (see [**10**], prop. 5.5.6, p. 220). The characteristic polynomial is then given by

$$P_n(x) = x^4 + A(n)x^3 + B(n)x^2 + A(n)x + 1,$$

where $A(n) = -(S_n + \bar{S_n})$, $B(n) = 2+ \mid S_n \mid^2$ with $S_n = \beta_n + \dfrac{1}{\beta_n}$.

Moreover, we have

$$\begin{array}{rcl} A(n) & = & -d_{22}\lambda^{-n} - \lambda^n a_{22} - n\nu c_{11} - a_{11} - d_{11}, \\ B(n) & = & \lambda^n \left[\mid A \mid + a_{22}d_{11} - c_{12}b_{21} + n\nu(a_{22}c_{11} - c_{12}a_{21}) \right] \\ & & +\lambda^{-n} \left[\mid D \mid + a_{11}d_{22} - c_{21}b_{12} + n\nu\Delta \right] \\ & & +(a_{11}d_{11} + a_{22}d_{22} - c_{22}b_{22} - c_{11}b_{11}). \end{array}$$

We must consider two cases: $d_{22} \neq 0$ and $d_{22} = 0$.

- If $d_{22} \neq 0$, i.e. we have for n sufficiently large $A(n) \sim -d_{22}\lambda^{-n}$. In the same way, as $\Delta \neq 0$ and $\nu \neq 0$, we obtain $B(n) \sim n\nu\Delta\lambda^{-n}$. We deduce that $\mathrm{Re}S_n \sim d_{22}\lambda^{-n}$ and $\mid S_n \mid^2 \sim d_{22}^2\lambda^{-2n}$. We also have $\mid S_n \mid^2 \sim n\nu\Delta\lambda^{-n}$ using the inequality on $B(n)$. We obtain a contradiction. As a consequence, all the eigenvalues are reals.

We then have eigenvalues $x_1(n)$, $x_2(n)$ and $1/x_1(n)$, $1/x_2(n)$, $x_1(n) \in \mathbb{R}$ and $x_2(n) \in \mathbb{R}$. We denote by $S_1(n) = x_1(n) + 1/x_1(n)$ and $S_2(n) = x_2(n) + 1/x_2(n)$. We have $A(n) = -(S_1(n) + S_2(n))$ and $B(n) = 2 + S_1(n)S_2(n)$, so $S_1(n)(A(n) + S_1(n)) = -S_1(n)S_2(n)$. As $A(n) \sim -d_{22}\lambda^{-n}$ and $B(n) \sim n\nu\Delta\lambda^{-n}$, we conclude that $S_1(n) \sim -nd_{22}^{-1}\Delta$, so $x_1(n) \sim -nd_{22}^{-1}\Delta$. Using $A(n)$, we obtain $S_2(n) \sim d_{22}\lambda^{-n}$, so $x_2(n) \sim d_{22}\lambda^{-n}$, which concludes the proof.

- If $d_{22} = 0$, we have $A(n) = O(n)$. As $B(n) \sim n\nu\Delta\lambda^{-n}$, this implies that $\mathrm{Im}(S_n) \neq 0$. If we denote $\beta(n) = \beta_1(n) + i\beta_2(n)$, $\beta_1(n), \beta_2(n) \in \mathbb{R}$, we have $\mathrm{Im}(S_n) = \beta_2(n)(1- \mid \beta(n) \mid^{-1})$. As $\mathrm{Im}(S_n) \neq 0$, we deduce that $\beta_2(n) \neq 0$ and $1- \mid \beta(n) \mid^{-1} \neq 0$, i.e. $\beta_2(n) \neq 0$ and $\mid \beta(n) \mid \neq 1$. The eigenvalues are then hyperbolic.

This concludes the proof. □

The behaviour of the eigenvalues can be also given when $d_{22} = 0$, but depends on several assumptions on the form of the homoclinic matrix which have not a direct geometrical meaning.

In some cases of interest, we can obtain a stronger result. For example, using the homoclinic matrix introduced in [**11**] and generalized in [**4**] in relation with the Arnold model [**1**], we obtain:

Theorem 3. — *Let H be a three degrees of freedom Hamiltonian systems possessing a 2 dimensional partially hyperbolic tori with a transversal homoclinic connection. We keep* *the notations of §.3. We assume that the homoclinic matrix has the form*

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \delta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{10}$$

and δ is a parameter.

Then, the matrix $\Pi D f_l^n$ is hyperbolic for n sufficiently large if and only if the matrix Π is transverse, i.e. $\delta \neq 0$ and the torus is with torsion, i.e. $\nu \neq 0$.

Proof. — The characteristic polynomial of $\Pi D f_l^n$ is given by

$$P(x) = (x^2 - x(\delta n \nu + 2) + 1)(x^2 - x a(n) + 1),$$

where $a(n) = \lambda^{2n} + \lambda^{-n}$. The matrix is hyperbolic if $\delta \neq 0$ and $\nu \neq 0$. Indeed, in this case, the matrix Π satisfies the transversality assumption. Moreover, if $\delta = 0$ and $\nu \neq 0$ (or $\delta \neq 0$ and $\nu = 0$), we obtain two eigenvalues equal to ± 1, destroying the hyperbolicity. This concludes the proof. □

4.3. Hyperbolicity of the homoclinic transition map. — We keep notations from section 3. We want to prove that under the assumptions of the transversality-torsion phenomenon the homoclinic transition map Δ is also hyperbolic for μ sufficiently small. In order to prove this, we must control the remainder of Δ with respect to Δ_l. This can be done assuming for example a special dependance of the remainder of f and Γ with respect to f_l and Γ_l. We denote $r_f(\phi, s, \rho, u) = f(\phi, s, \rho, u) - f_l(\phi, s, \rho, u)$ and $r_\Gamma(z) = \Gamma(p^- + z) - \Gamma_l(p^- + z)$. We denote by $z = (z_\phi, z_s, z_\rho, z_u)$ the coordinates in C. We make the following assumptions, already used in ([**5**],assumption (h_3),p.273):

- (r_1) We have $r_f(\phi, s, \rho, u) = O_2(\rho, su)$.
- (r_2) We have $r_\Gamma(z_\phi, z_s, z_\rho, z_u) = O_2(z_\rho, z_s z_u)$.

These two assumptions must be seen as the counterpart, in the Poincaré section, of the special form of the remainder in the Eliasson's normal form (1) for the Hamiltonian. We then have the following result:

Theorem 4. — *Let H be a 3 degrees of freedom Hamiltonian system possessing a 2 dimensional partially hyperbolic tori with a transversal homoclinic connection. We keep notations from section 3. We assume that assumptions (r_1) and (r_2) are satisfied, and that:*

- *i) The homoclinic matrix Π is strongly transverse;*
- *ii) The torus is with torsion.*

Then, for μ sufficiently small the homoclinic transition map Δ is hyperbolic.

This follows from the following results, already proved in ([**5**],p.290). We denote by $R(z) = \Delta(z) - \Delta_l(z)$ the remainder of the homoclinic transition map.

Lemma 3. — *(Control lemma) For each $z \in D_n$, we have $\| R(z) \| < C\lambda^{2n}$ and $\| DR(z) \| < \tilde{C}\mu\lambda^n$, where C and $\tilde{C}$ are constants.*

As a consequence, for each $z \in D_n$, the maps Δ and Δ_l are $\mu\lambda^n$ close in C^1 topology, as long as n is sufficiently large in order to have $\lambda^n \leq \mu$. Using classical perturbation theory for hyperbolic maps this concludes the proof.

Remark 1. — *In ([**6**],p.243) R.W. Easton assumes that f is linear in the Poincaré section. As a consequence, the maps Δ reduces to $\Delta = (W_\mu)^{-1} \circ \Gamma \circ \psi_l \circ W_\mu$. In this case, Δ_l approaches Δ in the C^1-topology when μ goes to zero (see [**6**],p.250). Indeed, this is equivalent to prove that Γ and Γ_l are C^1-close in a sufficiently small neighbourhood of the homoclinic orbit.*

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