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Strict Solutions of Nonlinear Hyperbolic Neutral Differential Equations

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Abstract—In this work, we study a class of abstract semilinear functional differential equations of a neutral type. Our main results concern the existence, uniqueness, and regularity of solutions. We assume that the linear part is nondensely defined, closed, and satisfies the Hille-Yosida condition.

Keywords—Abstract semilinear functional differential equations of neutral type, Hille-Yosida condition, Integrated semigroup, Integral solution, Strict solution.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we study the abstract Cauchy problem

$$\frac{d}{dt}\left(x(t) - Lx_t\right) = A_0x(t) + F(t, x_t), \qquad \text{for } t \geq 0, \\ x_0 = \varphi \in C_E, \tag{1}$$

where $A_0 : D(A_0) \subset E \to E$ is a linear operator, E is a Banach space, $C_E := C([-r,0],E)$; $r > 0$, L is a continuous linear operator from C_E into E, F is a function from $[0,T] \times C_E$, $T > 0$ into E, and for $x \in C([-r,b],E)$, $b > 0$, and $t \in [0,b]$, the function x_t denotes the element of C_E defined by $x_t(\theta) = x(t+\theta)$, for $\theta \in [-r,0]$. It is well known that if A_0 is the infinitesimal generator of a C_0-semigroup of bounded linear operators in E and under some conditions on F and L (see, for example, [1,2]), the classical semigroup theory ensures the well posedness of problem (1).

In [3], we considered system (1) in the case where F is linear, $L : \varphi \to B\varphi(-r)$, with $B \in \mathcal{L}(E)$ and A_0 satisfying the usual Hille-Yosida conditions except the density of $D(A_0)$ into E (see Definition 4). A natural generalized notion of the solution was provided by the integral solutions. The basic existence and uniqueness result was given and the solution was shown to generate an integrated semigroup. In this paper, we assume the following.

(H$_1$) A_0 satisfies the Hille-Yosida condition on E (without being densely defined).

(H$_2$) $L : C_E \to E$ is defined by $L\varphi = L_0\varphi + \sum_{j=1}^{n} B_j\varphi(-h_j)$, for all $\varphi \in C_E$, where range $(L) \subseteq D(A_0)$, L_0 is a continuous linear operator from C_E into E such that $\|L_0\| < 1$, $0 < h_1 < \cdots < h_n = r$ are given real numbers, and $B_j \in \mathcal{L}(E)$, $j = 1, 2, \ldots, n$.

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(H_3) F is a continuous function from $[0,T] \times C_E$, $T > 0$, into E and there exists a positive constant K_0 such that $\|F(t,\varphi_1) - F(t,\varphi_2)\| \le K_0\|\varphi_1 - \varphi_2\|$, for $t \in [0,T]$, $\varphi_1, \varphi_2 \in C_E$.

There are many examples in concrete situations where evolution equations are not densely defined. Only hypothesis (H_1) holds. One can refer to [3] or [4] for more details and [5] for another approach in the case of partial F.D.E. Nondensity occurs, in many situations, from restrictions made on the space where the equation is considered (for example, periodic continuous functions, Hölder continuous functions) or from boundary conditions (e.g., the space C^1 with null value on the boundary is nondense in the space of continuous functions).

Our objective in this paper is to prove existence, uniqueness, and regularity of solutions and to give a natural generalization of previous related results [1,3,6].

We now give a short review of the theory of integrated semigroups.

DEFINITION 1. *(See [7].) Let X be a Banach space. A family $(S(t))_{t\ge 0} \subset \mathcal{L}(X)$ is called an integrated semigroup if the following conditions are satisfied:*

(i) *$S(0) = 0$;*
(ii) *for any $x \in X$, $S(t)x$ is a continuous function of $t \ge 0$ with values in X; and*
(iii) *for any $t, s \ge 0$, $S(s)S(t) = \int_0^s (S(t+\tau) - S(\tau))\,d\tau$.*

DEFINITION 2. *(See [7].) An operator A is called a generator of an integrated semigroup, if there exists $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$, and there exists a strongly continuous exponentially bounded family $(S(t))_{t\ge 0}$ of linear bounded operators such that $S(0) = 0$ and $(\lambda I - A)^{-1} = \lambda \int_0^{+\infty} e^{-\lambda t} S(t)\,dt$ for all $\lambda > \omega$.*

DEFINITION 3. *(See [8].) An integrated semigroup $(S(t))_{t\ge 0}$ is called locally Lipschitz continuous, if for all $\tau > 0$, there exists a constant $k(\tau) > 0$ such that*

$$\|S(t) - S(s)\| \le k(\tau)|t - s|, \qquad \text{for all } t, s \in [0,\tau].$$

In this case, we know from [8], that $(S(t))_{t\ge 0}$ is exponentially bounded.

DEFINITION 4. *(See [8].) We say that a linear operator A satisfies the Hille-Yosida condition* (HY) *if there exist $M \ge 0$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and*

$$\sup\left\{(\lambda - \omega)^n \left\|(\lambda I - A)^{-n}\right\|,\, n \in \mathbf{N},\, \lambda > \omega\right\} \le M. \tag{HY}$$

THEOREM 5. *(See [8].) The following assertions are equivalent.*

(i) *A is the generator of a locally Lipschitz continuous integrated semigroup.*
(ii) *A satisfies the condition* (HY).

2. MAIN RESULTS

DEFINITION 6. *We say that a function $u : [-r,T] \to E$ is an integral solution of equation (1) if the following conditions hold:*

(i) *$u \in C([-r,T],E)$;*
(ii) *$\int_0^t u(s)\,ds \in D(A_0)$, for $t \in [0,T]$; and*
(iii) *$u(t) = Lu_t + \varphi(0) - L\varphi + A_0(\int_0^t u(s)\,ds) + \int_0^t F(s,u_s)\,ds$, for $0 \le t \le T$, $u_0 = \varphi$.*

DEFINITION 7. *We say that a function $x : [-r,T] \to E$ is a strict solution of equation (1) if*

(i) *$x(t) - Lx_t \in C^1([0,T],E) \cap C([0,T],D(A_0))$,*
(ii) *x satisfies equation (1) on $[0,T]$, and*
(iii) *$x_0 = \varphi$.*

REMARK 1.

(1) Theorem 5 shows that A_0 is the generator of a locally Lipschitz continuous integrated semigroup $(S_0(t))_{t\geq 0}$ on E and we have $\|S_0(t)\| \leq M_0 e^{\omega_0 t}$, for $t \geq 0$.
(2) The condition (H$_2$) implies that $A_0 L$ is a bounded linear operator on C_E.
(3) If u is an integral solution of equation (1), then for all $t \in [0,T]$, $u(t) \in \overline{D(A_0)}$, in particular $\varphi(0) \in \overline{D(A_0)}$.
(4) If x is an integral solution of (1) such that $t \to x(t) - Lx_t \in C^1([0,T],E)$ or $C([0,T], D(A_0))$, then x is a strict solution of (1).

Now we state the first result.

THEOREM 8. *Assume that* (H$_1$), (H$_2$), *and* (H$_3$) *hold. Let* $\varphi \in C_E$ *such that* $\varphi(0) \in \overline{D(A_0)}$. *Then, equation (1) has a unique integral solution* u *given by*

$$u(t) = Lu_t + S_0'(t)(\varphi(0) - L\varphi) + \frac{d}{dt}\int_0^t S_0(t-s)\left((A_0L)u_s + F(s,u_s)\right) ds, \qquad \text{for } t \in [0,T],$$

where $(S_0(t))_{t\geq 0}$ *is the integrated semigroup on* E *generated by* A_0.

For proving this theorem, we need the following general result.

LEMMA 9. *Let* $(U(t))_{t\geq 0}$ *be a locally Lipschitz continuous integrated semigroup on a Banach space* E *and* $G : [0,T] \to E\,(T > 0)$, *a Bochner-integrable function. Then, the function* $K : [0,T] \to E$ *defined by* $K(t) = \int_0^t U(t-s)G(s)\,ds$ *is continuously differentiable on* $[0,T]$ *and satisfies* $|\frac{dK}{dt}(t)|_E \leq 2k\int_0^t |G(s)|_E\,ds$, *for* $t \in [0,T]$, *where* $k := k(T)$ *is the Lipschitz constant of* $U(\cdot)$ *on* $[0,T]$.

PROOF OF LEMMA 9. It is immediate that, for $t, \tau \in [0,T]$,

$$K(t) - K(\tau) = \int_\tau^t U(t-s)G(s)\,ds + \int_0^\tau (U(t-s) - U(\tau-s))G(s)\,ds.$$

This gives

$$|K(t) - K(\tau)|_E \leq 2k\,|t-\tau| \int_0^{\max(t,\tau)} |G(s)|_E\,ds.$$

Consequently, K is Lipschitz continuous on $[0,T]$. Since $C^1([0,T],E)$ is dense in $L^1(0,T,E)$ with respect to the L^1-norm, it follows that there exists a sequence $(G_n)_{n\in\mathbf{N}}$, $G_n \in C^1([0,T],E)$ which converges to G in $L^1(0,T,E)$.

For each $n \in \mathbf{N}$, let K_n be the function defined by $K_n(t) = \int_0^t U(s)G_n(t-s)\,ds$, for $t \in [0,T]$. This shows that $K_n \in C^1([0,T],E)$, for $n \in \mathbf{N}$, and satisfies

$$|K_n - K|_{\mathrm{Lip}} \leq 2k\,|G_n - G|_{L^1}, \qquad \text{for } n \in \mathbf{N}.$$

But, $C^1([0,T],E)$ is a closed subspace of $\mathrm{Lip}([0,T],E)$. We conclude that $K \in C^1([0,T],E)$, and satisfies

$$\left|\frac{dK}{dt}(t)\right|_E \leq 2k\int_0^t |G(s)|_E\,ds, \qquad \text{for } t \in [0,T]. \qquad \blacksquare$$

PROOF OF THEOREM 8. It follows immediately form (H$_2$) that equation (1) can be written as

$$\begin{aligned} \frac{d}{dt}\left(x(t) - Lx_t\right) &= A_0\left(x(t) - Lx_t\right) + (A_0L)\,x_t + F(t,x_t), \qquad \text{for } t \geq 0, \\ x_0 &= \varphi \in C_E. \end{aligned} \tag{2}$$

In conjunction with this system, we consider an integrated form given by

$$x(t) = Lx_t + S_0'(t)\left(\varphi(0) - L\varphi\right) + \frac{d}{dt}\int_0^t S_0(t-s)\left((A_0L)\,x_s + F(s,x_s)\right) ds, \tag{3}$$

for $t \geq 0$, and $x(t) = \varphi(t)$, for $t \in [-r, 0]$. Let $0 < T_0 \leq T$. Consider the set

$$S_\varphi = \{x \in C([-r, T_0], E) : x(\theta) = \varphi(\theta), \text{ for } \theta \in [-r, 0]\}.$$

S_φ is a closed subset of $C([-r, T_0], E)$. Let H be the operator defined on $C([-r, T_0], E)$ by

$$H(x)(t) = Lx_t + S_0'(t)(\varphi(0) - L\varphi) + \frac{d}{dt}\int_0^t S_0(t-s)\left((A_0L)\,x_s + F(s, x_s)\right) ds,$$
$$\text{if } t \in [0, T_0] \text{ and } H(x)(t) = \varphi(t) \text{ if } t \in [-r, 0].$$

It follows immediately that $H(S_\varphi) \subset S_\varphi$ and for $x, y \in S_\varphi$, $t \in [0, T_0]$,

$$(Hx - Hy)(t) = L(x_t - y_t) + \frac{d}{dt}\int_0^t S_0(t-s)\left((A_0L)(x_s - y_s)(F(s, x_s) - F(s, y_s))\right) ds.$$

First, we choose $T_0 \leq h_1$. Then, we obtain $L(x_t - y_t) = L_0(x_t - y_t)$, for $t \in [0, T_0]$. By Lemma 9, we deduce that, for $t \in [0, T_0]$,

$$(Hx - Hy)(t) = L_0(x_t - y_t) + \frac{d}{dt}\int_0^t S_0(t-s)\left((A_0L)(x_s - y_s) + (F(s, x_s) - F(s, y_s))\right) ds.$$

So, we obtain

$$|(Hx - Hy)(t)| \leq \left[\|L_0\| + 2kT_0\left(\|A_0L\| + K_0\right)\right]\|x - y\|, \qquad \text{for } t \in [0, T_0],$$

where $\|x - y\|$ denotes the supnorm on $[-r, T_0]$ and $k := k(T)$ is the Lipschitz constant of $S_0(\cdot)$ on $[0, T]$. Let T_0 be a constant such that $\|L_0\| + 2kT_0(\|A_0L\| + K_0) < 1$ and $0 < T_0 \leq h_1$. Then, H is a strict contraction in S_φ. So, H has one and only one fixed point u in S_φ. We conclude that equation (1) has one and only one integral solution which is defined on the interval $[-r, T_0]$ and satisfies

$$u(t) = Lu_t + S_0'(t)(\varphi(0) - L\varphi) + \frac{d}{dt}\int_0^t S_0(t-s)\left((A_0L)\,u_s + F(s, u_s)\right) ds.$$

If $T_0 = T$, the proof is complete. If this is not the case, we can repeat the previous argument on $[-r, T_1]$, where $T_1 = \min\{T_0 + h_1, T\}$, with the initial condition $x(t) = u(t)$, for $t \in [T_0 - r, T_0]$. In this case, it is easy to see that, for $t \in [T_0, T_1]$, we have

$$x(t) = Lx_t + S_0'(t - T_0)(u(t) - Lu_t) + \frac{d}{dt}\int_{T_0}^t S_0(t-s)\left((A_0L)\,x_s + F(s, x_s)\right) ds.$$

We also have

$$L(x_t - y_t) = L_0(x_t - y_t), \qquad \text{for } t \in [0, T_1].$$

We remark that $\|L_0\| + 2k(T_1 - T_0)(\|A_0L\| + K_0) < 1$. We obtain in such a way an integral solution on $[-r, T_1]$.

If $T_1 < T$, we can repeat the previous argument. At the end, we obtain an integral solution of problem (1) defined on $[-r, T]$. ∎

THEOREM 10. *Assume that* (H$_1$), (H$_2$), *and* (H$_3$) *hold,* $F : [0, T] \times C_E \to E$ *is continuously differentiable, and there exist constants* K_1, $K_2 \geq 0$ *such that*

$$\|D_tF(t, \varphi) - D_tF(t, \psi)\| \leq K_1\|\varphi - \psi\| \text{ and } \|D_\varphi F(t, \varphi) - D_\varphi F(t, \psi)\| \leq K_2\|\varphi - \psi\|,$$
$$\text{for all } t \in [0, T] \text{ and } \varphi, \psi \in C_E,$$

where D_tF and $D_\varphi F$ denote the derivatives. Then, for given $\varphi \in C_E$ such that $\varphi(0) \in D(A_0)$, $\varphi' \in C_E$, $\varphi'(0) \in \overline{D(A_0)}$, and $\varphi'(0) = L\varphi' + A_0\varphi(0) + F(0,\varphi)$, the integral solution $u : [-r,T] \to E$ of equation (1), such that $u_0 = \varphi$ is the unique strict solution of equation (1).

PROOF. We know from Theorem 8 that equation (1) has a unique integral solution u which is given, for $0 \le t \le T$, by

$$u(t) = Lu_t + S_0'(t)(\varphi(0) - L\varphi) + \frac{d}{dt}\int_0^t S_0(t-s)((A_0L)u_s + F(s,u_s))\,ds. \tag{4}$$

Since $\varphi(0) \in D(A_0)$, then $\varphi(0) - L\varphi \in D(A_0)$ and we deduce that

$$S_0'(t)(\varphi(0) - L\varphi) = \varphi(0) - L\varphi + S_0(t)(A_0\varphi(0) - (A_0L)\varphi).$$

So, u can be written as

$$u(t) = Lu_t + \varphi(0) - L\varphi + S_0(t)(A_0\varphi(0) - (A_0L)\varphi) + \frac{d}{dt}\int_0^t S_0(t-s)((A_0L)u_s + F(s,u_s))\,ds.$$

Consider the following Cauchy problem:

$$\begin{aligned} \frac{d}{dt}(y(t) - Ly_t) &= A_0y(t) + D_tF(t,u_t) + D_\varphi F(t,u_t)y_t, \qquad \text{for } 0 \le t \le T,\\ y_0 &= \varphi'. \end{aligned} \tag{5}$$

By assumptions on φ, we know that $\varphi' \in C_E$ and $\varphi'(0) \in \overline{D(A_0)}$. Theorem 8 implies that equation (5) has a unique integral solution y which is given by $y_0 = \varphi'$ and for $t \in [0,T]$

$$\begin{aligned} y(t) &= Ly_t + S_0'(t)(\varphi'(0) - L\varphi')\\ &\quad + \frac{d}{dt}\int_0^t S_0(t-s)((A_0L)y_s + D_tF(s,u_s) + D_\varphi F(s,u_s)y_s)\,ds. \end{aligned} \tag{6}$$

Let $w : [-r,T] \to E$ be the function defined by

$$w(t) = \begin{cases} \varphi(0) + \int_0^t y(s)\,ds, & \text{for } 0 \le t \le T,\\ \varphi(t), & \text{for } -r \le t \le 0. \end{cases} \tag{7}$$

We will show that $u = w$ on $[0,T]$.

Using the expression (6) and the expressions satisfied by φ, we obtain, for $0 \le t \le T$,

$$\begin{aligned} w(t) &= L(w_t) + \varphi(0) - L\varphi + S_0(t)(A_0\varphi(0) + F(0,\varphi))\\ &\quad + \int_0^t S_0(t-s)((A_0L)y_s + D_tF(s,u_s) + D_\varphi F(s,u_s)y_s)\,ds. \end{aligned} \tag{8}$$

It follows that, for $0 \le t \le T$,

$$\begin{aligned} S_0(t)((A_0L)\varphi + F(0,\varphi)) &= \frac{d}{dt}\int_0^t S_0(t-s)((A_0L)w_s + F(s,w_s))\,ds\\ &\quad - \int_0^t S_0(t-s)((A_0L)y_s + D_tF(s,w_s) + D_\varphi F(s,w_s)y_s)\,ds. \end{aligned}$$

Consider the functions z_1 and z_2 defined on $[0,T]$ by

$$z_1(t) = u(t) - Lu_t \quad \text{and} \quad z_2(t) = w(t) - Lw_t.$$

First, we take $t \in [0, h_1]$. We obtain

$$z_1(t) - z_2(t) = \frac{d}{dt}\int_0^t S_0(t-s)\left((A_0L)(u_s - w_s) + F(s,u_s) - F(s,w_s)\right) ds$$
$$- \int_0^t S_0(t-s)\left(D_tF(s,u_s) - D_tF(s,w_s)\right) ds$$
$$- \int_0^t S_0(t-s)\left(D_\varphi F(s,u_s) - D_\varphi F(s,w_s)\right) y_s\, ds.$$

If we put

$$c = \max\left(\sup_{0\le s\le T}\|S_0(s)\|,\ \sup_{-r\le s\le T}\sup|y(s)|\right),$$

we obtain $|z_1(t) - z_2(t)| \le \gamma\int_0^t \|u_s - w_s\|\, ds$, where $\gamma = 2k(\|A_0L\| + K_0) + cK_1 + c^2K_2$. On the other hand, we have $u = w$ in $[-r, 0]$ and $L(u_t - w_t) = L_0(u_t - w_t)$, for $t \in [0, h_1]$. Then, we obtain $\|u_t - w_t\| \le \|L_0\|\|u_t - w_t\| + \gamma\int_0^t \|u_s - w_s\|\, ds$. The condition $\|L_0\| < 1$ implies that $\|u_t - w_t\| \le (\gamma/(1 - \|L_0\|))\int_0^t \|u_s - w_s\|\, ds$. By Gronwall's inequality, we conclude that $u_t = w_t$, for $t \in [0, h_1]$.

Repeating the same procedure in $[h_1, 2h_1]$, $[2h_1, 3h_1], \dots, [ph_1, (p+1)h_1]$, we obtain that $u = w$ on $[-r, T]$. So, $t \to u(t) - Lu_t$ is continuously differentiable on $[0, T]$ and u is the unique strict solution of (1) on $[-r, T]$. ∎

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