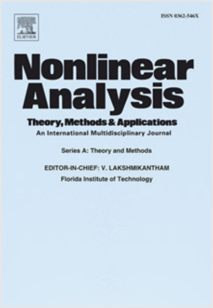

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Reduction Principle and Dynamic Behaviors for a Class of Partial Functional Differential Equations [1]

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Abstract: We investigate the dynamic of solutions in the $\alpha-$norm for some nonhomogeneous linear partial functional differential equations. We suppose that the undelayed homogeneous part is the infinitesimal generator of an analytic semigroup, the delayed part is continuous with respect to fractional powers of the generator. We establish a reduction principle for the infinite dimensional system in order to reduce its qualitative analysis to a finite dimensional one. Our reduction method is based on a new variation of constants formula. As application, the reduced system is used to prove the existence of almost automorphic, almost periodic and periodic solutions for the whole infinite dimensional system.

Keywords and phrases: Analytic semigroup; Fractional power of operators; Mild solution; Variation of constants formula; Spectral decomposition; Reduction principle; Almost automorphic solution.

AMS (MOS) Subject Classifications: 34k30, 34K14, 34K20, 35B15, 35B35.

1. Introduction

The reduction principle for infinite dynamical systems is an interesting tool to reduce their complexity. Frequently, a model formulated in terms of partial differential equations (PDE) can be reduced to an ordinary differential equation (ODE) in finite dimensional space. This ODE is usually obtained by projecting the PDE on a finite dimensional subspace which retains most relevant features of the whole system. At any rate the asymptotic behavior of the original system is the same as that of the reduced one.

In this work, we study the behavior of the following class of partial functional differential equations

$$\begin{cases} \dfrac{d}{dt}u(t) = -Au(t) + L(u_t) + f(t) \quad \text{for } t \geq \sigma, \\ u_\sigma = \varphi \in \mathcal{C}_\alpha := \mathcal{C}([-r, 0]\,; X_\alpha), \end{cases} \tag{1.1}$$

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where the operator $-A : D(A) \to X$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space $(X, |.|)$. For $0 < \alpha \leq 1$, X_α denotes the Banach space $D(A^\alpha)$ endowed with the norm

$$|x|_\alpha = |A^\alpha x| \text{ for } x \in D(A^\alpha),$$

where A^α is the fractional power of A which will be defined below. $\mathcal{C}_\alpha := \mathcal{C}([-r, 0]\,; X_\alpha)$ is the space of continuous functions from $[-r, 0]$ to X_α endowed with the uniform norm topology

$$|\phi|_{\mathcal{C}_\alpha} = \sup_{-r\leq\theta\leq 0} |\phi(\theta)|_\alpha.$$

As usual, for every $\sigma \in \mathbb{R}$ and $t \geq \sigma$, the history function $u_t \in \mathcal{C}_\alpha$ is defined by

$$u_t(\theta) = u(t + \theta) \text{ for } \theta \in [-r, 0]\,.$$

L is a bounded linear operator from $\mathcal{C}_\alpha$ into X and f is a continuous function from $[\sigma, +\infty)$ into X. As a model for the class (1.1) one may take the following system which describes many phenomena in physical systems

$$\begin{cases} \dfrac{\partial}{\partial t}v(t, x) = \sum_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial}{\partial x_j}v(t, x)\right) - a_0v(t, x) + \varepsilon \sum_{i=1}^{n} \dfrac{\partial}{\partial x_i}v(t - r_i, x) \\ \qquad\qquad + \displaystyle\int_{-r_{n+1}}^{0} \beta(\theta)v(t + \theta, x)d\theta + \Theta\,(t, x) \text{ for } t \geq 0 \text{ and } x \in \Omega, \\ v(t, x) = 0 \text{ for } t \geq 0 \text{ and } x \in \partial\Omega, \\ v(\theta, x) = \varphi_0(\theta, x) \text{ for } \theta \in [-r, 0] \text{ and } x \in \Omega, \end{cases} \tag{1.2}$$

where a_0, ε are constants, $r = \max\{r_1, ..., r_{n+1}\}$, Ω is an open bounded set in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $\beta \in L^2([-r_{n+1}, 0]\,, \mathbb{R})$, $\Theta : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is continuous, the initial function $\varphi_0 : [-r, 0] \times \Omega \to \mathbb{R}$ is a given function.

The theory and applications of partial functional differential equations are an active research area. It has been extensively studied in the past years (see [1], [2], [3], [4], [5], [20], [28] and the references therein). Additionally, such equations can exhibit a very rich behavior like almost periodicity, almost automorphy or periodicity (see [7] and [8]). Since 1974, Travis and Webb [25], [26], [27] established the basic theory for the existence and stability of solutions for the following equation

$$\begin{cases} \dfrac{d}{dt}u(t) = -Au(t) + G(t, u_t) \text{ for } t \geq \sigma, \\ u_\sigma = \varphi \in \mathcal{C}_\alpha := \mathcal{C}([-r, 0]\,; X_\alpha), \end{cases}$$

where $G : [\sigma, +\infty) \times X \to X$ is a nonlinear continuous function, as application the authors proposed in [26] the following model

$$\begin{cases} \dfrac{\partial}{\partial t}v(t, x) = \dfrac{\partial^2}{\partial x^2}v(t, x) + F\left(v(t - r, x), \dfrac{\partial}{\partial x}v(t - r, x)\right) \text{ for } t \geq 0 \text{ and } x \in [0, \pi]\,, \\ v(t, 0) = v(t, \pi) = 0 \text{ for } t \geq 0, \\ v(\theta, x) = \varphi_0(\theta, x) \text{ for } \theta \in [-r, 0] \text{ and } x \in [0, \pi]\,, \end{cases}$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz continuous function.

In this paper, we develop a new reduction principle for Equation (1.1) to prove that the dynamic of bounded solutions on $\mathbb{R}$ is governed by an ordinary differential equation in a finite dimensional space. This goal will be done through a new variation of constants formula. As application, we propose to study the existence of almost automorphic, almost periodic and periodic solutions of Equation (1.1). In the periodic case, usually we use the fixed point theory to prove that the Poincaré map associated to the equation has a fixed point. In the almost automorphic and the almost periodic cases the situation is different and more complicated since the fixed point approach cannot be applied.

Almost automorphic, almost periodic and periodic solutions are interesting phenomena in dynamical systems. Recall that the concept of almost automorphy is more general than the one of almost periodicity. It was introduced by Bochner and studied by many authors. For more details on almost automorphic functions we refer to [23]. Let us recall some well established results in this field. Consider the following ordinary differential equation

$$\frac{d}{dt}x(t) = Bx(t) + b(t) \quad \text{for } t \in \mathbb{R}, \tag{1.3}$$

where B is a constant $n \times n$-matrix and $b : \mathbb{R} \to \mathbb{R}^n$ is continuous and τ-periodic. In [19], Massera proved the existence of an τ-periodic solution under the existence of a bounded solution on $\mathbb{R}^+$. Bohr and Neugebauer extended Massera's Theorem to almost periodic case (see [12]). Recently, the results in [18] extended Bohr and Neugebauer's Theorem to almost automorphic case for Equation (1.3). For partial functional differential equations, we refer to [1], [3], [4], [7], [8], [10] and [11].

For Equation (1.1), we use the reduction principle to show the existence of almost automorphic, almost periodic and periodic solutions. Massera's Theorem, Bohr and Neugebauer's Theorem and the results of [18] are extended to Equation (1.1): the existence of a bounded solution on $\mathbb{R}^+$ implies the existence of an almost automorphic (resp. almost periodic, resp. periodic) solution if the imput function f is almost automorphic (resp. almost periodic, resp. periodic).

The organization of this work is as follows: in Section 2, we recall some preliminary results on the fractional powers of unbounded linear operators generating analytic semigroups and some results regarding the existence of solutions for Equation (1.1). In Section 3, we establish a variation of constants formula for Equation (1.1). In Section 4, we develop a reduction principle for Equation (1.1). In Section 5, we prove the existence of almost automorphic, almost periodic and periodic solutions of Equation (1.1). In the hyperbolic case, we study the uniqueness of almost automorphic, almost periodic and periodic solutions of Equation (1.1). In the last section, we apply our theoretical results to the partial functional differential equation (1.2).

2. **Preliminary results**

We recall some results about fractional powers that will be used in the next. Here and hereafter we assume that

($\mathbf{H}_0$) $-A : D(A) \to X$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space $(X, |.|)$ such that

$$|T(t)x| \leq Me^{\omega t}|x| \quad \text{for } \; t \geq 0 \; \text{ and } x \in X,$$

where $M \geq 1$ and $\omega \in \mathbb{R}$. Without loss of generality, we assume that $0 \in \rho(-A)$. Otherwise we can substitute $-A + \delta I$ to $-A$ such that $0 \in \rho(-A + \delta I)$.

It is well known (see [24, p. 69-70]) that Condition $(\mathbf{H_0})$ allows us to define, for $0 < \alpha \leq 1$, the linear operator $A^{-\alpha}$ on X by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} T(t) dt$$

where Γ denotes the Gamma function

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\alpha t} dt.$$

Definition 2.1. [24, p. 74] For $0 < \alpha \leq 1$, the fractional power A^{α} of A is the closed linear operator defined from $D(A^{\alpha}) = Im(A^{-\alpha})$ to X by

$$A^{\alpha} = (A^{-\alpha})^{-1}.$$

Theorem 2.2. [24, p. 74] *The following properties hold.*
(**i**) *$A^{-\alpha}$ is a bounded linear operator on X.*
(**ii**) *$(X_{\alpha}, |.|_{\alpha})$ is a Banach space.*
(**iii**) *If $0 < \beta < \alpha \leq 1$, then $X_{\alpha} \hookrightarrow X_{\beta}$.*
(**iv**) *$T(t) : X \to X_{\alpha}$ for every $t > 0$.*
(**v**) *For every $t > 0$, the operator $A^{\alpha}T(t)$ is bounded on X and*

$$|A^{\alpha}T(t)x| \leq M_{\alpha} \frac{e^{\omega t}}{t^{\alpha}} |x| \quad \text{for } x \in X \text{ and } t > 0, \tag{2.1}$$

where M_{α} is a positive constant and $\omega \in \mathbb{R}$ is given by $(\mathbf{H_0})$.

We consider the following definition of the mild solutions for Equation (1.1).

Definition 2.3. [26] Let $(\sigma, \varphi) \in \mathbb{R} \times \mathcal{C}_{\alpha}$. A function $u : [\sigma - r, +\infty) \to X_{\alpha}$ is said to be a mild solution of Equation (1.1) if the following conditions hold
(**i**) $u : [\sigma - r, +\infty) \to X_{\alpha}$ is continuous,
(**ii**) $u(t) = T(t-\sigma)\varphi(0) + \int_{\sigma}^{t} T(t-s) \left[L(u_s) + f(s) \right] ds$ for $t \geq \sigma$,
(**iii**) $u_{\sigma} = \varphi$.

Theorem 2.4. [26] *For all $\varphi \in \mathcal{C}_{\alpha}$, Equation (1.1) has a unique mild solution on $[\sigma - r, +\infty)$.*

Throughout this work, the mild solutions of Equation (1.1) are denoted by $u(., \sigma, \varphi, f)$ and will be called solutions of Equation (1.1). For any $t \geq 0$, we define the operator $\mathcal{U}(t)$ on $\mathcal{C}_{\alpha}$ by

$$\mathcal{U}(t)\varphi = u_t(., 0, \varphi, 0) \quad \text{for } \varphi \in \mathcal{C}_{\alpha}.$$

Proposition 2.5. [26] $(\mathcal{U}(t))_{t \geq 0}$ *is a strongly continuous semigroup on $\mathcal{C}_{\alpha}$, that is:*
(**i**) *for all $t \geq 0$, $\mathcal{U}(t)$ is a bounded linear operator on $\mathcal{C}_{\alpha}$,*
(**ii**) *$\mathcal{U}(0) = I$,*
(**iii**) *$\mathcal{U}(t+s) = \mathcal{U}(t)\mathcal{U}(s)$ for all $t, s \geq 0$,*
(**iv**) *for all $\varphi \in \mathcal{C}_{\alpha}$, $\mathcal{U}(t)\varphi$ is a continuous function of $t \geq 0$ with values in $\mathcal{C}_{\alpha}$.*

Proposition 2.6. [26] *Let $\mathcal{A}_{\mathcal{U}}$ be the infinitesimal generator of $(\mathcal{U}(t))_{t\geq 0}$. Then*

$$\begin{cases} D(\mathcal{A}_{\mathcal{U}}) = \{\varphi \in \mathcal{C}_\alpha : \varphi' \in \mathcal{C}_\alpha,\ \varphi(0) \in D(A) \text{ and } \varphi(0) = -A\varphi(0) + L(\varphi)\}, \\ \\ \mathcal{A}_{\mathcal{U}}\varphi = \varphi'. \end{cases}$$

We assume that

$(\mathbf{H_1})$ $T(t)$ is compact on X for each $t > 0$.

Consequently, we have the following important result on the eventual compactness of the semigroup $(\mathcal{U}(t))_{t\geq 0}$.

Proposition 2.7. [27, Proposition 4.1] *Assume that $(\mathbf{H_1})$ holds. Then for each $t > r$, $\mathcal{U}(t)$ is compact on $\mathcal{C}_\alpha$.*

3. Variation of constants formula for Equation (1.1)

In this section, we obtain a variation of constants formula associated to the nonhomogeneous Equation (1.1) in terms of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ in the space $\mathcal{C}_\alpha := \mathcal{C}([-r, 0]; X_\alpha)$. This formula is inspired by [17], where the authors considered an infinite delay but for $\alpha = 0$. We used our new variation of constants formula to explain some asymptotic behaviors of solutions of Equation (1.1) in the α-norm.

Without loss of generality, we assume that $\omega > 0$ and we introduce for $n > n_0 = \left[\max\left(\omega, \frac{1}{r}\right)\right] + 1$, the function $\Lambda^n x$ defined, for $x \in X$, by

$$(\Lambda^n x)(\theta) = \begin{cases} (n\theta + 1)B_n x & \text{if } \theta \in [-\frac{1}{n}, 0], \\ \\ 0 & \text{if } \theta \in [-r, -\frac{1}{n}], \end{cases}$$

where $[y]$ denotes the integer part of y and $B_n = nR(n, -A)$ with $R(n, -A) = (nI + A)^{-1}$. It follows from (**iii**) of Theorem 2.2 that for any $x \in X$ the function $\Lambda^n x$ belongs to $\mathcal{C}_\alpha$. The following theorem gives a variation of constants formula for Equation (1.1) in $\mathcal{C}_\alpha$.

Theorem 3.1. *The solution $u(., \sigma, \varphi, f)$ of Equation (1.1) satisfies the following variation of constants formula*

$$u_t(., \sigma, \varphi, f) = \mathcal{U}(t - \sigma)\varphi + \lim_{n\to+\infty} \int_\sigma^t \mathcal{U}(t - s)\Lambda^n f(s)ds \quad \text{for } t \geq \sigma. \tag{3.1}$$

Moreover, the above limit exists uniformly over $t \geq \sigma$ such that $t - \sigma$ is bounded.

For the proof, we need the following Proposition.

Lemma 3.2. [15, Lemma 2] *Let ψ and $w : [a, b] \to [0, \infty)$ be continuous functions. If $\Phi(\cdot)$ is nondecreasing and there are constants $\vartheta > 0$ and $0 < \alpha < 1$ such that*

$$\Psi(t) \leq \Phi(t) + \vartheta \int_a^b \frac{\Psi(s)}{(t - s)^{1-\alpha}} ds \text{ for } t \in [a, b].$$

Then for every $t \in [a, b]$ and every $k \in \mathbb{N}$ such that $k\alpha > 1$, we have the following estimation

$$\Psi(t) \leq e^{\vartheta^k \Gamma(\alpha)^k (t-a)^{k\alpha}/\Gamma(k\alpha)} \sum_{i=0}^{k-1} \left(\frac{\vartheta(b - a)}{\alpha}\right)^i \Phi(t).$$

Proof of Theorem 3.1 . Let $u(.,\sigma,\varphi,f)$ be the solution of Equation (1.1). Then for $t\geq\sigma$, we have

$$\begin{aligned} u_t(.,\sigma,\varphi,f) &= u_t(.,\sigma,\varphi,0)+u_t(.,\sigma,0,f),\\ &= \mathcal{U}(t-\sigma)\varphi+u_t(.,\sigma,0,f). \end{aligned}$$

We claim in the $\mathcal{C}_\alpha$-norm that

$$\lim_{n\to+\infty}\int_\sigma^t \mathcal{U}(t-s)\Lambda^n f(s)ds = u_t(.,\sigma,0,f),$$

uniformly in (t,σ) such that $t\geq\sigma$ and $t-\sigma$ is bounded. In fact, let

$$g^n(t,\sigma)=\int_\sigma^t \mathcal{U}(t-s)\Lambda^n f(s)ds \;\text{ for } t\geq\sigma.$$

It is enough to prove formula (3.1) in $[\sigma,\sigma+r]$ and then we proceed by steps. Let $\sigma\leq t\leq\sigma+r$. Then

$$g^n(t,\sigma)(\theta)=\int_\sigma^t (\mathcal{U}(t-s)\Lambda^n f(s))(\theta)ds=\int_\sigma^t u(t+\theta-s,0,\Lambda^n f(s),0)ds.$$

On the other hand,

$$u(t+\theta-s,0,\Lambda^n f(s),0)=\begin{cases} T(t+\theta-s)B_nf(s)+\displaystyle\int_s^{t+\theta}T(t+\theta-\xi)L(\mathcal{U}(\xi-s)\Lambda^n f(s))d\xi \\ \hfill \text{if }\; s\leq t+\theta, \\ (n(t+\theta-s)+1)\,B_nf(s) \;\text{ if }\; t+\theta\leq s\leq t+\theta+\frac{1}{n}, \\ 0 \;\text{ if } s\geq t+\theta+\frac{1}{n}. \end{cases}$$

Then

$$g^n(t,\sigma)(\theta)=\begin{cases} \displaystyle\int_\sigma^{\min\{t,t+\theta+\frac{1}{n}\}}(n(t+\theta-s)+1)\,B_nf(s)ds \;\text{ if } \theta\in[-(t-\sigma)-\frac{1}{n},-(t-\sigma)], \\ 0 \;\text{ if } \theta\in[-r,-(t-\sigma)-\frac{1}{n}]. \end{cases}$$

Let $-(t-\sigma)\leq\theta\leq 0$. Then

$$g^n(t,\sigma)(\theta)=\int_\sigma^{t+\theta}u(t+\theta-s,0,\Lambda^n f(s),0)ds+\int_{t+\theta}^t u(t+\theta-s,0,\Lambda^n f(s),0)ds.$$

Therefore for $-(t-\sigma)\leq\theta\leq 0$

$$\begin{aligned} g^n(t,\sigma)(\theta) &= \int_\sigma^{t+\theta} T(t+\theta-s)B_nf(s)ds + \int_\sigma^{t+\theta}\int_s^{t+\theta} T(t+\theta-\xi)L(\mathcal{U}(\xi-s)\Lambda^n f(s))d\xi ds \\ &\quad + \int_{t+\theta}^{\min\{t,t+\theta+\frac{1}{n}\}} \left(n(t+\theta-s)+1\right)B_nf(s)ds, \\ &= \int_\sigma^{t+\theta} T(t+\theta-s)B_nf(s)ds + \int_\sigma^{t+\theta}\int_\sigma^{\xi} T(t+\theta-\xi)L(\mathcal{U}(\xi-s)\Lambda^n f(s))dsd\xi \\ &\quad + \int_{t+\theta}^{\min\{t,t+\theta+\frac{1}{n}\}} \left(n(t+\theta-s)+1\right)B_nf(s)ds, \end{aligned}$$

$$\begin{aligned} &= \int_\sigma^{t+\theta} T(t+\theta-s)B_nf(s)ds + \int_\sigma^{t+\theta} T(t+\theta-\xi)L\left(\int_\sigma^{\xi}\mathcal{U}(\xi-s)\Lambda^n f(s)ds\right)d\xi \\ &\quad + \int_{t+\theta}^{\min\{t,t+\theta+\frac{1}{n}\}} \left(n(t+\theta-s)+1\right)B_nf(s)ds, \\ &= \int_\sigma^{t+\theta} T(t+\theta-s)B_nf(s)ds + \int_\sigma^{t+\theta} T(t+\theta-s)L\left(g^n(s,\sigma)\right)ds \\ &\quad + \int_{t+\theta}^{\min\{t,t+\theta+\frac{1}{n}\}} \left(n(t+\theta-s)+1\right)B_nf(s)ds. \end{aligned}$$

Finally, we get the following expression of $g^n(t,\sigma)(\theta)$ for $\theta\in[-r,0]$

$$g^n(t,\sigma)(\theta) = \begin{cases} \displaystyle\int_\sigma^{t+\theta} T(t+\theta-s)B_nf(s)ds + \int_\sigma^{t+\theta} T(t+\theta-s)L\left(g^n(s,\sigma)\right)ds \\ \displaystyle\qquad + \int_{t+\theta}^{\min\{t,t+\theta+\frac{1}{n}\}} \left(n(t+\theta-s)+1\right)B_nf(s)ds \quad \text{if } \theta\in[-(t-\sigma),0], \\ \displaystyle\int_\sigma^{\min\{t,t+\theta+\frac{1}{n}\}} \left(n(t+\theta-s)+1\right)B_nf(s)ds \quad \text{if } \theta\in[-(t-\sigma)-\tfrac{1}{n},-(t-\sigma)], \\ 0 \quad \text{if } \theta\in[-r,-(t-\sigma)-\tfrac{1}{n}]. \end{cases}$$

Let $m>n>n_0$ and

$$W^{n,m}(t,\sigma) = g^n(t,\sigma) - g^m(t,\sigma).$$

If $-(t-\sigma) \leq \theta \leq 0$, we get

$$\begin{aligned} W^{n,m}(t,\sigma)(\theta) \quad &= \int_{\sigma}^{t+\theta} T(t+\theta-s)L\left(W^{n,m}(s,\sigma)\right)ds \\ &+ \int_{\sigma}^{t+\theta} T(t+\theta-s)(B_n-B_m)f(s)ds \\ &+ \int_{t+\theta}^{\min\{t,t+\theta+\frac{1}{n}\}} \left(n(t+\theta-s)+1\right)B_n f(s)ds \\ &- \int_{t+\theta}^{\min\{t,t+\theta+\frac{1}{m}\}} \left(m(t+\theta-s)+1\right)B_m f(s)ds. \end{aligned}$$

If $-(t-\sigma)-\frac{1}{m} \leq \theta \leq -(t-\sigma)$, we obtain

$$\begin{aligned} W^{n,m}(t,\sigma)(\theta) = &\int_{\sigma}^{\min\{t,t+\theta+\frac{1}{n}\}} \left(n(t+\theta-s)+1\right)B_n f(s)ds \\ &- \int_{\sigma}^{\min\{t,t+\theta+\frac{1}{m}\}} \left(m(t+\theta-s)+1\right)B_m f(s)ds. \end{aligned}$$

If $-(t-\sigma)-\frac{1}{n} \leq \theta \leq -(t-\sigma)-\frac{1}{m}$, then

$$W^{n,m}(t,\sigma)(\theta) = \int_{\sigma}^{\min\{t,t+\theta+\frac{1}{n}\}} \left(n(t+\theta-s)+1\right)B_n f(s)ds.$$

If $-r \leq \theta \leq -(t-\sigma)-\frac{1}{n}$, then

$$W^{n,m}(t,\sigma)(\theta) = 0.$$

Since for $x \in X$

$$A^{\alpha}R(n,-A)x = \int_0^{+\infty} e^{-nt}A^{\alpha}T(t)xdt,$$

it follows from (**v**) of Theorem 2.2 that

$$|R(n,-A)x|_{\alpha} \leq M_{\alpha}\int_0^{+\infty} \frac{e^{-(n-\omega)t}}{t^{\alpha}}dt\ |x|.$$

Using the following formula for the Gamma function

$$\Gamma(1-\alpha)z^{\alpha-1} = \int_0^{+\infty} \frac{e^{-zt}}{t^{\alpha}}dt \quad \text{for } z>0, \tag{3.2}$$

we obtain

$$|B_n x|_{\alpha} \leq \frac{n}{(n-\omega)^{1-\alpha}}M_{\alpha}\Gamma(1-\alpha)|x|.$$

Consequently, we get

$$\sup_{\theta\in[-r,\ -(t-\sigma)]} |W^{n,m}(t,\sigma)(\theta)|_{\alpha} \leq \left(\frac{1}{(n-\omega)^{1-\alpha}}+\frac{1}{(m-\omega)^{1-\alpha}}\right)M_{\alpha}\Gamma(1-\alpha)\sup_{s\in[\sigma,\ t]}|f(s)|. \tag{3.3}$$

By computation, one can prove for $-(t-\sigma) \leq \theta \leq 0$, that

$$\begin{aligned} |W^{n,m}(t,\sigma)(\theta)|_\alpha &\leq \left(\frac{1}{(n-\omega)^{1-\alpha}} + \frac{1}{(m-\omega)^{1-\alpha}}\right) M_\alpha \Gamma(1-\alpha) \sup_{s\in[\sigma,\ t]} |f(s)| \\ &+M_\alpha \frac{e^{\omega(t-\sigma)}}{1-\alpha}(t-\sigma)^{1-\alpha} \sup_{s\in[\sigma,\ t]} |B_n f(s) - B_m f(s)| \\ &+M_\alpha |L| \int_\sigma^{t+\theta} \frac{e^{\omega(t+\theta-s)}}{(t+\theta-s)^\alpha} |W^{n,m}(s,\sigma)|_{\mathcal{C}_\alpha} ds. \end{aligned}$$

It follows that

$$\begin{aligned} |W^{n,m}(t,\sigma)(\theta)|_\alpha &\leq \left(\frac{1}{(n-\omega)^{1-\alpha}} + \frac{1}{(m-\omega)^{1-\alpha}}\right) M_\alpha \Gamma(1-\alpha) \sup_{s\in[\sigma,\ t]} |f(s)| \\ &+M_\alpha \frac{e^{\omega(t-\sigma)}}{1-\alpha}(t-\sigma)^{1-\alpha} \sup_{s\in[\sigma,\ t]} |B_n f(s) - B_m f(s)| \\ &+M_\alpha |L| \int_0^{t+\theta-\sigma} \frac{e^{\omega s}}{s^\alpha} |W^{n,m}(t+\theta-s,\sigma)|_{\mathcal{C}_\alpha} ds \end{aligned}$$

Let $m > n > n_0$ and $\mu^{n,m}(t) = \sup_{s\in[\sigma,t]} |W^{n,m}(s,\sigma)|_{\mathcal{C}_\alpha}$. Since $\mu^{n,m}(\cdot)$ is nondecreasing, then

$$\begin{aligned} |W^{n,m}(t,\sigma)|_\alpha &\leq \left(\frac{1}{(n-\omega)^{1-\alpha}} + \frac{1}{(m-\omega)^{1-\alpha}}\right) M_\alpha \Gamma(1-\alpha) \sup_{s\in[\sigma,\ t]} |f(s)| \\ &+M_\alpha \frac{e^{\omega(t-\sigma)}}{1-\alpha}(t-\sigma)^{1-\alpha} \sup_{s\in[\sigma,\ t]} |B_n f(s) - B_m f(s)| \\ &+M_\alpha |L| \int_0^{t-\sigma} \frac{e^{\omega s}}{s^\alpha} |\mu^{n,m}(t-s)| ds \end{aligned}$$

and

$$\begin{aligned} \mu^{n,m}(t) &\leq \left(\frac{1}{(n-\omega)^{1-\alpha}} + \frac{1}{(m-\omega)^{1-\alpha}}\right) M_\alpha \Gamma(1-\alpha) \sup_{s\in[\sigma,\ t]} |f(s)| \\ &+M_\alpha \frac{e^{\omega(t-\sigma)}}{1-\alpha}(t-\sigma)^{1-\alpha} \sup_{s\in[\sigma,\ t]} |B_n f(s) - B_m f(s)| \\ &+M_\alpha |L| \int_\sigma^{t} \frac{e^{\omega(t-s)}}{(t-s)^\alpha} \mu^{n,m}(s) ds. \end{aligned}$$

Then, for $\sigma \leq t \leq \sigma + r$

$$\begin{aligned}\mu^{n,m}(t) &\le \left(\frac{1}{(n-\omega)^{1-\alpha}}+\frac{1}{(m-\omega)^{1-\alpha}}\right)M_\alpha\Gamma(1-\alpha)\sup_{s\in[\sigma,\ t]}|f(s)|\\ &+M_\alpha\frac{e^{\omega(t-\sigma)}}{1-\alpha}(t-\sigma)^{1-\alpha}\sup_{s\in[\sigma,\ t]}|B_nf(s)-B_mf(s)|\\ &+M_\alpha|L|e^{\omega r}\int_\sigma^t\frac{\mu^{n,m}(s)}{(t-s)^\alpha}ds.\end{aligned}$$

Lemma 3.2 implies that for $\sigma\le t\le\sigma+r$

$$\begin{aligned}\mu^{n,m}(t)\le\Bigg[&\left(\frac{1}{(n-\omega)^{1-\alpha}}+\frac{1}{(m-\omega)^{1-\alpha}}\right)\Gamma(1-\alpha)\sup_{s\in[\sigma,t]}|f(s)|\\ &+\frac{e^{\omega(t-\sigma)}}{1-\alpha}(t-\sigma)^{1-\alpha}\sup_{s\in[\sigma,\ t]}|B_nf(s)-B_mf(s)|\Bigg]\\ &\times e^{(M_\alpha|L|e^{\omega r})^k\Gamma(1-\alpha)^k(t-\sigma)^{k(1-\alpha)}/\Gamma(k(1-\alpha))}\sum_{i=0}^{k-1}\left(\frac{M_\alpha|L|e^{\omega r}r}{1-\alpha}\right)^i,\end{aligned}$$

for every $k\in\mathbb{N}$ such that $k(1-\alpha)>1$. Then, for $\sigma\le t\le\sigma+r$, we have

$$\begin{aligned}|g^n(t,\sigma)-g^m(t,\sigma)|_{\mathcal{C}_\alpha}\le\Bigg[&\left(\frac{1}{(n-\omega)^{1-\alpha}}+\frac{1}{(m-\omega)^{1-\alpha}}\right)\Gamma(1-\alpha)\sup_{s\in[\sigma,t]}|f(s)|\\ &+\frac{e^{\omega(t-\sigma)}}{1-\alpha}(t-\sigma)^{1-\alpha}\sup_{s\in[\sigma,\ t]}|B_nf(s)-B_mf(s)|\Bigg]\\ &\times e^{(M_\alpha|L|e^{\omega r})^k\Gamma(1-\alpha)^k(t-\sigma)^{k(1-\alpha)}/\Gamma(k(1-\alpha))}\sum_{i=0}^{k-1}\left(\frac{M_\alpha|L|e^{\omega r}r}{1-\alpha}\right)^i,\end{aligned}$$

for every $k\in\mathbb{N}$ such that $k(1-\alpha)>1$. We conclude that $(g^n(t,\sigma))_{n\ge n_0}$ is a Cauchy sequence in $\mathcal{C}_\alpha$ uniformly in $[\sigma,\sigma+r]$. We denote by $g(t,\sigma)$ the limit of the sequence $(g^n(t,\sigma))_{n>n_0}$ in $\mathcal{C}_\alpha$, and we define the function h in $[\sigma,\sigma+r]$ by

$$h(t,\sigma)(\theta)=\begin{cases}\displaystyle\int_\sigma^{t+\theta}T(t+\theta-s)\left[f(s)+L\left(g(s,\sigma)\right)\right]ds & \text{if}\quad \theta\in[-(t-\sigma),0],\\ 0\ \text{ if }\ \theta\in[-r,-(t-\sigma)]\,. & \end{cases}$$

Then

$$\begin{aligned}|g^n(t,\sigma)-h(t,\sigma)|_{\mathcal{C}_\alpha}\le\frac{M_\alpha}{(n-\omega)^{1-\alpha}}\Gamma(1-\alpha)\sup_{s\in[\sigma,\ t]}|f(s)|+\Bigg[&|L|\sup_{s\in[\sigma,\ t]}|g^n(s,\sigma)-g(s,\sigma)|_{\mathcal{C}_\alpha}\\ &+\sup_{s\in[\sigma,\ t]}|B_nf(s)-f(s)|\Bigg]\frac{e^{\omega(t-\sigma)}}{1-\alpha}M_\alpha(t-\sigma)^{1-\alpha}.\end{aligned}$$

This implies that the sequence $(g^n(t,\sigma))_{n\geq n_0}$ converges to $h(t,\sigma)$ in $\mathcal{C}_\alpha$. Consequently,

$$g(t,\sigma)=h(t,\sigma) \quad \text{for } \sigma\leq t\leq \sigma+r.$$

Define the function $v:[\sigma-r,\sigma+r]\to X_\alpha$ by

$$v(t)=\begin{cases} g(t,\sigma)(0) & \text{if} \quad \sigma\leq t\leq\sigma+r,\\ 0 \ \text{ if } \ \sigma-r\leq t\leq\sigma. \end{cases}$$

Then

$$v_\sigma=0 \text{ and } v_t=g(t,\sigma) \quad \text{for } \sigma\leq t\leq\sigma+r.$$

It follows, for $\sigma\leq t\leq\sigma+r$,

$$\begin{aligned} v(t) &= g(t,\sigma)(0)\\ &= \int_\sigma^t T(t-s)\left[f(s)+L\left(g(s,\sigma)\right)\right]ds\\ &= \int_\sigma^t T(t-s)\left[f(s)+L(v_s)\right]ds\\ &= u(t,\ \sigma,\ 0,\ f). \end{aligned}$$

Consequently,

$$\lim_{n\to+\infty}\int_\sigma^t \mathcal{U}(t-s)\Lambda^n f(s)ds=g(t,\sigma)=v_t=u_t(.,\sigma,0,f), \tag{3.4}$$

uniformly in $[\sigma,\sigma+r]$. By steps, we use the same argument as above to prove the formula (3.4) for all $t\geq\sigma$. $\square$

4. **Reduction principle**

Here and hereafter we assume that $(\mathbf{H_1})$ holds. As a consequence of the compactness property of the semigroup $(\mathcal{U}(t))_{t\geq0}$, the spectrum $\sigma(\mathcal{A}_\mathcal{U})$ of the infinitesimal generator $\mathcal{A}_\mathcal{U}$ coincides with the point spectrum and we have

$$\sigma(\mathcal{A}_\mathcal{U})=\left\{\lambda\in\mathbb{C}:\ker\Delta(\lambda)\neq\{0\}\right\},$$

where the linear operator $\Delta(\lambda):X_\alpha\to X_\alpha$ is defined for $\lambda\in\sigma(\mathcal{A}_\mathcal{U})$ and $x\in X_\alpha$ by

$$\Delta(\lambda)x=\lambda x+Ax-L(e^{\lambda\cdot}x).$$

The domain $D(\Delta(\lambda))$ is given by

$$D(\Delta(\lambda))=\{x\in X_\alpha:x\in D(A) \text{ and } Ax-L(e^{\lambda\cdot}x)\in X_\alpha\}.$$

For $x\in X_\alpha$ and $\lambda\in\mathbb{C}$, the function $e^{\lambda\cdot}x\in\mathcal{C}_\alpha$ is defined by

$$\left(e^{\lambda\cdot}x\right)(\theta)=e^{\lambda\theta}x \quad \text{for } \theta\in[-r,0].$$

By Proposition 2.7 and [9, Theorem 5.3.7, p. 333], we obtain the following spectral decomposition of the phase space $\mathcal{C}_\alpha$.

Theorem 4.1. *$\mathcal{C}_\alpha$ is decomposed as follows*

$$\mathcal{C}_\alpha=S\oplus V,$$

where S is $\mathcal{U}$-invariant space and there are positive constants η and N such that

$$\left|\mathcal{U}(t)\varphi\right|_{\mathcal{C}_\alpha}\leq Ne^{-\eta t}\left|\varphi\right|_{\mathcal{C}_\alpha} \quad \textit{for } t\geq0 \textit{ and } \varphi\in S,$$

V is a finite dimensional space and the restriction of $\mathcal{U}$ to V is a group.

Let $\mathcal{C}_\alpha^*$ be the dual space of $\mathcal{C}_\alpha$ and $d = \dim(V)$ with a basis of V given by $\Phi = (\phi_1, ..., \phi_d)$. Then there exist d elements $\psi_1, ..., \psi_d$ in $\mathcal{C}_\alpha^*$ such that

$$\begin{cases} \langle \psi_i, \phi_j \rangle = \delta_{ij}, \text{ for } i, j = 1, ..., d, \\ \langle \psi_i, \phi \rangle = 0, \text{ for all } \phi \in S \text{ and } i = 1, ..., d, \end{cases} \tag{4.1}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and $\langle ., . \rangle$ denotes the duality pairing between $\mathcal{C}_\alpha^*$ and $\mathcal{C}_\alpha$. Let $\Psi = col\,(\psi_1, ..., \psi_d)$. Then $\langle \Psi, \Phi \rangle$ is a $d \times d$-matrix where its (i, j)-component is $\langle \psi_i, \phi_j \rangle$. Denote by Π^s and Π^v the projections respectively on S and V, and by $\mathcal{U}^s(t)$ and $\mathcal{U}^v(t)$ the restrictions of $\mathcal{U}(t)$ respectively on S and V. For each $\varphi \in \mathcal{C}_\alpha$, we have

$$\Pi^v \varphi = \Phi \langle \Psi, \varphi \rangle .$$

In fact, for $\varphi \in \mathcal{C}_\alpha$, we have $\varphi = \Pi^s \varphi + \Pi^v \varphi$ with $\Pi^v \varphi = \sum_{i=1}^{d} \alpha_i \phi_i$ and $\alpha_i \in \mathbb{R}$. By (4.1), we conclude that

$$\alpha_i = \langle \psi_i, \varphi \rangle .$$

Hence

$$\Pi^v \varphi = \sum_{i=1}^{d} \langle \psi_i, \varphi \rangle \phi_i = \Phi \langle \Psi, \varphi \rangle .$$

Since $(\mathcal{U}^v(t))_{t \in \mathbb{R}}$ is a group on V, then there exists a $d \times d$-matrix G such that

$$\mathcal{U}^v(t)\Phi = \Phi e^{tG} \quad \text{for } t \in \mathbb{R}.$$

Moreover, $\sigma(G) = \{\lambda \in \sigma(\mathcal{A}_\mathcal{U}) : Re(\lambda) \geq 0\}$ where $\mathcal{A}_\mathcal{U}$ is the infinitesimal generator of $(\mathcal{U}(t))_{t \geq 0}$. For $i \in \{1, ..., d\}$, we define the linear mapping $x_{i,n}^*$ by

$$x_{i,n}^*(a) = \langle \psi_i, \Lambda^n a \rangle \quad \text{for } n \geq n_0 \text{ and } a \in X.$$

Define the d-column vector $x_n^* = col\left(x_{1,n}^*, ..., x_{d,n}^*\right)$. Then

$$\langle x_n^*, a \rangle = \langle \Psi, \Lambda^n a \rangle \text{ for } a \in X$$

where for $i = 1, ..., d$,

$$\langle x_n^*, a \rangle_i = \langle \psi_i, \Lambda^n a \rangle .$$

Let $\mathcal{L}(X, \mathbb{R}^d)$ denote the space of bounded linear operators from X to $\mathbb{R}^d$.

Theorem 4.2. *There exists $x^* \in \mathcal{L}(X, \mathbb{R}^d)$ such that $(x_n^*)_{n \geq n_0}$ converges to x^* in the weak sense*

$$\langle x_n^*, x \rangle \underset{n \to +\infty}{\longrightarrow} \langle x^*, x \rangle \quad \textit{for } x \in X.$$

For the proof of Theorem 4.2, we need the following lemma.

Lemma 4.3. *Let $u_t(., \sigma, 0, f)$ be the solution of Equation (1.1) with $\varphi = 0$ and $t \geq \sigma$. Then*

$$\Pi^v u_t(., \sigma, 0, f) = \Phi \lim_{n \to +\infty} \int_\sigma^t e^{(t-\xi)G} \langle x_n^*, f(\xi) \rangle \, d\xi.$$

Proof of Lemma 4.3. The solution $u_t(.,\sigma,0,f)$ of Equation (1.1) with $\varphi=0$ is given by

$$u_t(.,\sigma,0,f)=\lim_{n\to+\infty}\int_\sigma^t \mathcal{U}\left(t-\xi\right)\left(\Lambda^n f(\xi)\right)d\xi.$$

Then

$$\Pi^v u_t(.,\sigma,0,f)=\lim_{n\to+\infty}\int_\sigma^t \mathcal{U}^v\left(t-\xi\right)\Pi^v\left(\Lambda^n f(\xi)\right)d\xi.$$

Since

$$\Pi^v\left(\Lambda^n f(\xi)\right)=\Phi\left\langle\Psi,\Lambda^n f(\xi)\right\rangle=\Phi\left\langle x_n^*,f(\xi)\right\rangle,$$

it follows that

$$\begin{aligned}\Pi^v u_t(.,\sigma,0,f)&=\Phi\lim_{n\to+\infty}\int_\sigma^t e^{(t-\xi)G}\left\langle\Psi,\Lambda^n f(\xi)\right\rangle d\xi,\\&=\Phi\lim_{n\to+\infty}\int_\sigma^t e^{(t-\xi)G}\left\langle x_n^*,f(\xi)\right\rangle d\xi.\ \square\end{aligned}$$

Proof of Theorem 4.2. Let $x\in X$ be fixed and consider Equation (1.1) with the constant function $f(.)=x$, $\sigma=0$ and $\varphi=0$. Then for $t\geq 0$

$$\Pi^v u_t(.,0,0,x)=\Phi\overline{z}(t,x),$$

where $\overline{z}(t,x)\in\mathbb{R}^d$. On the other hand, by Lemma 4.3

$$\Pi^v u_t(.,0,0,x)=\Phi\lim_{n\to+\infty}\left(\int_0^t e^{\xi G}d\xi\right)\left\langle x_n^*,x\right\rangle.$$

It follows that

$$\lim_{n\to+\infty}\left(\int_0^t e^{\xi G}d\xi\left\langle x_n^*,x\right\rangle\right)=\overline{z}(t,x). \tag{4.2}$$

Let $t_0>0$ be such that $\displaystyle\int_0^{t_0} e^{\xi G}d\xi$ is invertible. Then

$$\lim_{n\to+\infty}\left\langle x_n^*,x\right\rangle=\left(\int_0^{t_0} e^{\xi G}d\xi\right)^{-1}\overline{z}(t_0,x)\quad\text{for } x\in X.$$

By Banach-Steinhaus's theorem, we deduce that there exists $x^*\in\mathcal{L}(X,\mathbb{R}^d)$ such that

$$\left\langle x_n^*,x\right\rangle\underset{n\to+\infty}{\longrightarrow}\left\langle x^*,x\right\rangle\quad\text{for } x\in X.\square$$

As a consequence of Theorem 4.2 we obtain that

Corollary 4.4. *For any continuous function $e:\mathbb{R}\to X$, the following formula holds for $t,\sigma\in\mathbb{R}$*

$$\lim_{n\to+\infty}\int_\sigma^t \mathcal{U}^v\left(t-\xi\right)\Pi^v\left(\Lambda^n e(\xi)\right)d\xi=\Phi\int_\sigma^t e^{(t-\xi)G}\left\langle x^*,e(\xi)\right\rangle d\xi.$$

Using the above corollary we get a new reduction principle.

Theorem 4.5. *(**Reduction Principle**). Let u be a solution of the following equation (1.1). Then, $z(t) = \langle \Psi, u_t \rangle$ is a solution of the following ordinary differential equation*

$$\frac{d}{dt} z(t) = Gz(t) + \langle x^*, f(t) \rangle \quad \text{for } t \geq \sigma. \tag{4.3}$$

Conversely, if f is a bounded function on $\mathbb{R}$ and z is a solution of Equation (4.3) on $\mathbb{R}$, then the function u given for $t \in \mathbb{R}$ by

$$u(t) = \left[\Phi z(t) + \lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-\xi) \Pi^s (\Lambda^n f(\xi)) \, d\xi \right](0)$$

is a solution of the following equation in the whole real line

$$\frac{d}{dt} u(t) = -Au(t) + L(u_t) + f(t) \quad \text{for } t \in \mathbb{R}. \tag{4.4}$$

Proof. Let u be a solution of Equation (1.1) on $\mathbb{R}$. Then for $t \geq \sigma$, we have

$$u_t = \Pi^s u_t + \Pi^v u_t$$

and

$$\Pi^v u_t = \mathcal{U}^v(t-\sigma) \Pi^v u_\sigma + \lim_{n \to +\infty} \int_{\sigma}^{t} \mathcal{U}^v(t-\xi) \Pi^v (\Lambda^n f(\xi)) \, d\xi.$$

Since $\Pi^v u_t = \Phi \langle \Psi, u_t \rangle$, then by Corollary 4.4 we get that

$$\begin{aligned} \Phi \langle \Psi, u_t \rangle &= \mathcal{U}^v(t-\sigma) \Phi \langle \Psi, u_\sigma \rangle + \Phi \int_{\sigma}^{t} e^{(t-\xi)G} \langle x^*, f(\xi) \rangle \, d\xi, \\ &= \Phi e^{(t-\sigma)G} \langle \Psi, u_\sigma \rangle + \Phi \int_{\sigma}^{t} e^{(t-\xi)G} \langle x^*, f(\xi) \rangle \, d\xi. \end{aligned}$$

Let $z(t) = \langle \Psi, u_t \rangle$. Then for $t \geq \sigma$

$$z(t) = e^{(t-\sigma)G} z(\sigma) + \int_{\sigma}^{t} e^{(t-\xi)G} \langle x^*, f(\xi) \rangle \, d\xi.$$

Consequently, z is a solution of the ordinary differential Equation (4.3) on $\mathbb{R}$. Conversely, assume that f is bounded on $\mathbb{R}$. Then,

$$\lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-\xi) \Pi^s (\Lambda^n f(\xi)) \, d\xi$$

is well defined on $\mathbb{R}$. Let z be a solution of (4.3) on $\mathbb{R}$ and consider the function v defined on $\mathbb{R}$ by

$$v(t) = \Phi z(t) + \lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-\xi) \Pi^s (\Lambda^n f(\xi)) \, d\xi.$$

Let $t, \sigma \in \mathbb{R}$. Since

$$z(t) = e^{(t-\sigma)G} z(\sigma) + \int_{\sigma}^{t} e^{(t-\xi)G} \langle x^*, f(\xi) \rangle \, d\xi,$$

and using Corollary 4.4, the function v_1 given by

$$v_1(t) = \Phi z(t)$$

satisfies

$$v_1(t) = \mathcal{U}^v\left(t-\sigma\right)v_1(\sigma) + \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}^v\left(t-\xi\right)\Pi^v\left(\Lambda^n f(\xi)\right)d\xi.$$

Moreover, the function v_2 given by

$$v_2(t) = \lim_{n\to+\infty}\int_{-\infty}^t \mathcal{U}^s\left(t-\xi\right)\Pi^s\left(\Lambda^n f(\xi)\right)d\xi,$$

satisfies for $t\geq\sigma$ the following expression

$$v_2(t) = \mathcal{U}^s\left(t-\sigma\right)v_2\left(\sigma\right) + \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}^s\left(t-\xi\right)\Pi^s\left(\Lambda^n f(\xi)\right)d\xi.$$

Let $t\geq\sigma$. Then

$$\begin{aligned}\mathcal{U}\left(t-\sigma\right)v\left(\sigma\right) &= \mathcal{U}^v\left(t-\sigma\right)v_1(\sigma) + \mathcal{U}^s\left(t-\sigma\right)v_2(\sigma),\\ &= v_1(t) - \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}^v\left(t-\xi\right)\Pi^v\left(\Lambda^n f(\xi)\right)d\xi\\ &\quad + v_2(t) - \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}^s\left(t-\xi\right)\Pi^s\left(\Lambda^n f(\xi)\right))d\xi,\\ &= v(t) - \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}\left(t-\xi\right)\left(\Lambda^n f(\xi)\right)d\xi.\end{aligned}$$

Therefore

$$v\left(t\right) = \mathcal{U}\left(t-\sigma\right)v\left(\sigma\right) + \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}\left(t-\xi\right)\left(\Lambda^n f(\xi)\right)d\xi.$$

By Theorem 3.1, we obtain that the function u defined by $u(t) = v\left(t\right)(0)$ is a solution of Equation (4.4) on $\mathbb{R}$. $\square$

5. **Almost automorphic solutions**

In this section, we study the existence of almost automorphic, almost periodic and periodic solutions of Equation (4.4). We need to recall some results on this topic. Let $\mathcal{BC}(\mathbb{R}, Y)$ be the space of all bounded continuous functions from $\mathbb{R}$ to Y provided with the uniform norm topology. For $k\in\mathcal{BC}(\mathbb{R},\,Y)$ and $\tau\in\mathbb{R}$, we define the function k_τ by

$$k_\tau(s) = k(\tau+s) \quad \text{for all } s\in\mathbb{R}.$$

Definition 5.1. [12] A continuous function $k:\mathbb{R}\to Y$ is said to be almost periodic if

$$\{k_\tau : \tau\in\mathbb{R}\} \text{ is relatively compact in } \mathcal{BC}(\mathbb{R}, Y).$$

Definition 5.2. [23] A continuous function $k:\mathbb{R}\to Y$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_n$ there exists a subsequence $(s_n)_n$ such that

$$\lim_{n\to\infty} k(t+s_n) = \chi(t) \text{ exists for all } t \text{ in } \mathbb{R}$$

and

$$\lim_{n\to\infty}\chi(t-s_n) = k(t) \quad \text{for all } t \text{ in } \mathbb{R}.$$

Remark 1. Because of the pointwise convergence, the function χ is just measurable but not necessarily continuous. Almost periodic functions are almost automorphic.
Remark 2. An almost automorphic function may not be uniformly continuous.
Example. Let $k : \mathbb{R} \to \mathbb{R}$ be defined by

$$k(t) = \sin\left(\frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)}\right).$$

Then, k is an almost automorphic function which is not uniformly continuous on $\mathbb{R}$.
Remark 2 and the above example are very important and indicate that many results and methods in the theory of almost periodicity may not stand in almost automorphy framework.
We recall the results about the existence of periodic, almost automorphic and almost periodic solutions for the ordinary differential equation (1.3).

Theorem 5.3. [19] *Assume that $b : \mathbb{R} \to \mathbb{R}^n$ is τ-periodic. Then the existence of a bounded solution on $\mathbb{R}^+$ of Equation (1.3) implies the existence of a τ-periodic solution of Equation (1.3).*

Theorem 5.4. [12, Theorem 5.8, p. 86] *Assume that $b : \mathbb{R} \to \mathbb{R}^n$ is an almost periodic function. Then the existence of a bounded solution on $\mathbb{R}^+$ of Equation (1.3) implies the existence of an almost periodic solution of Equation (1.3).*

Combining Theorem 3.1 and Lemma 3.3 in [18], we obtain the following theorem which gives the relationship between the existence of bounded solution on $\mathbb{R}^+$ and the existence of almost automorphic solutions of ordinary differential equation (1.3).

Theorem 5.5. [18] *Assume that $b : \mathbb{R} \to \mathbb{R}^n$ is an almost automorphic function. If Equation (1.3) has a bounded solution on $\mathbb{R}^+$, then it has an almost automorphic solution.*

Now, we state and prove a theorem on the existence of almost automorphic, almost periodic and periodic solutions for Equation (4.4).

Theorem 5.6. *The existence of a bounded solution on $\mathbb{R}^+$ of Equation (1.1) implies the existence of an almost automorphic (resp. almost periodic, resp. τ-periodic) solution of Equation (4.4) if f is almost automorphic (resp. almost periodic, resp. τ-periodic).*

Proof. Let u be a bounded solution of Equation (1.1) on $\mathbb{R}^+$. By Theorem 4.5, the function $z(t) = \langle \Psi, u_t \rangle$ is a bounded solution on $\mathbb{R}^+$ of the ordinary differential equation (4.3). Moreover, the function $\varkappa$ defined on $\mathbb{R}$ by

$$\varkappa(t) = \langle x^*, f\,(t) \rangle$$

is almost periodic (resp. almost automorphic) from $\mathbb{R}$ to $\mathbb{R}^d$ if f is almost periodic (resp. almost automorphic). By Theorem 5.4 (resp. Theorem 5.5), we get that the reduced system (4.3) has an almost periodic (resp. almost automorphic) solution $\widetilde{z}$. Consequently, $\Phi\widetilde{z}(.)$ is an almost periodic (resp. almost automorphic) function on $\mathbb{R}$. By Theorem 4.5, the function $u(t) = v(t)(0)$ such that

$$v(t) = \Phi\widetilde{z}\,(t) + \lim_{n\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s\,(t-\xi)\,\Pi^s\,(\Lambda^n f(\xi))\,d\xi,$$

is a solution of Equation (4.4) on $\mathbb{R}$. To end the proof, we show that the function defined on $\mathbb{R}$ by

$$y(t)=\lim_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^{s}\left(t-\xi\right)\Pi^{s}\left(\Lambda^{n}f(\xi)\right)d\xi$$

is almost periodic (resp. almost automorphic). In fact, if f is an almost periodic function then for any sequence of real numbers $\left(\alpha_{p}^{'}\right)_{p\geq0}$ there exists a subsequence $\left(\alpha_{p}\right)_{p\geq0}\subset\left(\alpha_{p}^{'}\right)_{p\geq0}$ such that $f(.+\alpha_{p})$ converges uniformly on $\mathbb{R}$ to some function $\widetilde{f}$ and $y(.+\alpha_{p})$ converges uniformly on $\mathbb{R}$ to the function

$$\widetilde{y}(t)=\lim_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^{s}\left(t-\xi\right)\Pi^{s}\left(\Lambda^{n}\widetilde{f}(\xi)\right)d\xi.$$

Consequently, y is an almost periodic function and v is an almost periodic solution of Equation (4.4). Assume that f is almost automorphic. Then for any sequence of real numbers $\left(\alpha_{p}^{'}\right)_{p\geq0}$ there exists a subsequence $\left(\alpha_{p}\right)_{p\geq0}$ of $\left(\alpha_{p}^{'}\right)_{p\geq0}$ such that for $t\in\mathbb{R}$

$$\lim_{p\to\infty}f(t+\alpha_{p})=\widetilde{h}(t)$$

and

$$\lim_{p\to\infty}\widetilde{h}(t-\alpha_{p})=f(t).$$

Then for $t\in\mathbb{R}$

$$y(t+\alpha_{p})=\lim_{n\to+\infty}\int_{-\infty}^{t+\alpha_{p}}\mathcal{U}^{s}\left(t+\alpha_{p}-\xi\right)\Pi^{s}\left(\Lambda^{n}f\left(\xi\right)\right)d\xi,$$

which gives that

$$y(t+\alpha_{p})=\lim_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^{s}\left(t-\xi\right)\Pi^{s}\left(\Lambda^{n}f\left(\xi+\alpha_{p}\right)\right)d\xi.$$

Lebesgue's dominated convergence theorem implies that

$$y(t+\alpha_{p})\to w(t)\text{ as }p\to\infty,$$

where w is a function defined for $t\in\mathbb{R}$ by

$$w(t)=\lim_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^{s}\left(t-\xi\right)\Pi^{s}\left(\Lambda^{n}\widetilde{h}\left(\xi\right)\right)d\xi.$$

Then for $t\in\mathbb{R}$

$$w(t-\alpha_{p})=\lim_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^{s}\left(t-\xi\right)\Pi^{s}\left(\Lambda^{n}\widetilde{h}\left(\xi-\alpha_{p}\right)\right)d\xi.$$

Using the same argument as above, we can prove that

$$w(t-\alpha_{p})\to y(t)\text{ as }p\to\infty.$$

Consequently, we conclude that y is an almost automorphic function and v is also an almost automorphic solution of Equation (4.4). The periodic case can be treated by using the same approach and Theorem 5.3. □

In the next, we investigate the existence and uniqueness of periodic, almost periodic and almost automorphic solutions in the case where the semigroup $\left(\mathcal{U}(t)\right)_{t\geq0}$ is hyperbolic.

Definition 5.7. We say that the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if

$$\sigma(\mathcal{A}_{\mathcal{U}}) \cap i\mathbb{R} = \emptyset.$$

The eventual compactness of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ implies the following result on the spectral decomposition of the phase space $\mathcal{C}_\alpha$.

Theorem 5.8. *Assume that the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic. Then the space $\mathcal{C}_\alpha$ is decomposed as a direct sum of two $\mathcal{U}$-invariant closed subspaces S and V such that the restriction of $(\mathcal{U}(t))_{t\geq 0}$ on V is a group and there exist positive constants N_0 and η such that*

$$\begin{cases} |\mathcal{U}(t)\varphi|_{\mathcal{C}_\alpha} \leq N_0 e^{-\eta t} |\varphi|_{\mathcal{C}_\alpha} & \text{for } t \geq 0 \text{ and } \varphi \in S, \\ |\mathcal{U}(t)\varphi|_{\mathcal{C}_\alpha} \leq N_0 e^{\eta t} |\varphi|_{\mathcal{C}_\alpha} & \text{for } t \leq 0 \text{ and } \varphi \in V. \end{cases}$$

As a consequence of the hyperbolicity we get the following theorem.

Theorem 5.9. *Assume that the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic. If f is bounded on $\mathbb{R}$, then Equation (4.4) has a unique bounded solution w on $\mathbb{R}$ which is given by*

$$w_t = \lim_{n\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-\xi)\,\Pi^s(\Lambda^n f(\xi))\,d\xi - \lim_{n\to+\infty}\int_{t}^{+\infty} \mathcal{U}^v(t-\xi)\,\Pi^v(\Lambda^n f(\xi))\,d\xi.$$

Moreover, w is almost automorphic (resp. almost periodic, resp. periodic) if f is almost automorphic (resp. almost periodic, resp. periodic).

Proof. The function w is well defined and bounded on $\mathbb{R}$. In fact, for every $t \in \mathbb{R}$ we have

$$|w(t)|_\alpha \quad\leq\quad \frac{N_0}{\eta}|f|\left(|\Pi^s| + |\Pi^v|\right),$$

where $|f| = \sup_{s\in\mathbb{R}} |f(s)|$. Moreover, for all $t \geq \sigma$ we have

$$\begin{aligned} w_t &= \lim_{n\to+\infty}\int_{-\infty}^{\sigma} \mathcal{U}^s(t-\xi)\,\Pi^s(\Lambda^n f(\xi))\,d\xi + \lim_{n\to+\infty}\int_{\sigma}^{t} \mathcal{U}^s(t-\xi)\,\Pi^s(\Lambda^n f(\xi))\,d\xi \\ &\quad - \lim_{n\to+\infty}\int_{t}^{\sigma} \mathcal{U}^v(t-\xi)\,\Pi^v(\Lambda^n f(\xi))\,d\xi - \lim_{n\to+\infty}\int_{\sigma}^{+\infty} \mathcal{U}^v(t-\xi)\,\Pi^v(\Lambda^n f(\xi))\,d\xi, \\ &= \mathcal{U}(t-\sigma)w_\sigma + \lim_{n\to+\infty}\int_{\sigma}^{t} \mathcal{U}(t-\xi)\,\Lambda^n f(\xi)d\xi. \end{aligned}$$

This means that w is a bounded solution on $\mathbb{R}$ of Equation (4.4). For uniqueness, assume that there exists another bounded solution $\widetilde{w}$ of Equation (4.4). Then for $t \geq \sigma$,

$$w_t - \widetilde{w}_t = \mathcal{U}(t-\sigma)\left(w_\sigma - \widetilde{w}_\sigma\right)$$

and

$$|\Pi^s(w_t - \widetilde{w}_t)|_{\mathcal{C}_\alpha} \leq N_0 e^{-\eta(t-\sigma)}|\Pi^s(w_\sigma - \widetilde{w}_\sigma)|_{\mathcal{C}_\alpha}.$$

Since $w_\sigma - \widetilde{w}_\sigma$ is bounded, we get by letting $\sigma \to -\infty$

$$\Pi^s(w_t - \widetilde{w}_t) = 0.$$

It follows that

$$w_t - \widetilde{w}_t \in V.$$

Since $\mathcal{U}^v(t)$ is a group, then for $t \leq \sigma$

$$w_t - \widetilde{w}_t = \mathcal{U}^v(t-\sigma)\left(w_\sigma - \widetilde{w}_\sigma\right).$$

Then

$$|w_t - \widetilde{w}_t|_{\mathcal{C}_\alpha} \leq N_0 e^{\eta(t-\sigma)} |w_\sigma - \widetilde{w}_\sigma|_{\mathcal{C}_\alpha}.$$

Letting $\sigma \to +\infty$, we get

$$w_t = \widetilde{w}_t \ \text{ for all } t \in \mathbb{R}.$$

This proves that Equation (4.4) has a unique bounded solution on $\mathbb{R}$. By Theorem 5.6, this unique bounded solution w is almost automorphic (resp. almost periodic, resp. periodic) if f is almost automorphic (resp. almost periodic, resp. periodic). $\square$

6. **Applications**

To apply our theoretical results, we consider the following model of partial differential equation with several delays

$$\begin{cases} \dfrac{\partial}{\partial t} v(t,x) = \sum_{i,j=1}^n \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial}{\partial x_j} v(t,x)\right) - a_0 v(t,x) + \varepsilon \sum_{i=1}^n \dfrac{\partial}{\partial x_i} v(t-r_i,x) \\ \qquad\qquad + \displaystyle\int_{-r_{n+1}}^0 \beta(\theta) v(t+\theta,x)d\theta + \Theta(t,x) \quad \text{for } t \geq \sigma \text{ and } x \in \Omega, \\ v(t,x) = 0 \quad \text{for } t \geq \sigma \text{ and } x \in \partial\Omega, \\ v(\sigma+\theta,x) = \varphi_0(\theta,x) \quad \text{for } \theta \in [-r,0] \ \text{ and } x \in \Omega, \end{cases} \tag{6.1}$$

where $\sigma \in \mathbb{R}$, a_0 and ε are positive constants, $r = \max\{r_1, \ldots.., r_{n+1}\}$, Ω is an open bounded set in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $\beta \in L^2([-r_{n+1}, 0];\mathbb{R})$, $\Theta : [\sigma, +\infty) \times \Omega \to \mathbb{R}$ is continuous, the initial data $\varphi_0 : [-r, 0] \times \Omega \to \mathbb{R}$ is defined in the next. The coefficients $a_{ij} \in L^\infty(\Omega)$ are symmetric and satisfy the ellipticity condition, namely,

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \text{ for } x \in \Omega \text{ and } \xi \in \mathbb{R}^n,$$

for a positive constant ν.

In order to rewrite Equation (6.1) in the abstract form, we introduce $X = L^2(\Omega)$ and we define the linear operator $A : D(A) \subset X \to X$ by

$$\begin{cases} D(A) = H^2(\Omega) \cap H_0^1(\Omega), \\ A = -\sum_{i,j=1}^n \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial}{\partial x_j}\right). \end{cases}$$

Lemma 6.1. [24, Theorem 7.3.6] *$-A$ is the infinitesimal generator of a compact analytic semigroup $(T(t))_{t\geq 0}$ on X. Moreover, the spectrum $\sigma(-A)$ is the point spectrum and $\sigma(-A) = \{\lambda_n : n \in \mathbb{N}\}$ where $\ldots < \lambda_{n+1} < \lambda_n < \ldots < \lambda_0 < 0$.*

Lemma 6.2. *$|\nabla|$ and $|.|_{\frac{1}{2}}$ are equivalent on $D(A^{\frac{1}{2}})$. More precisely, we have*

$$\sqrt{\nu}\,|\nabla\psi| \le |\psi|_{\frac{1}{2}} \le \sqrt{\max_{1\le i,j\le n}|a_{i,j}|_{L^\infty}}\,|\nabla\psi|\,,$$

where ∇ is the gradient vector.

Proof. Since the coefficients a_{ij} are symmetric, then the operator A is self-adjoint. Then for $\psi \in D(A^{\frac{1}{2}})$

$$(A\psi,\psi) = (A^{\frac{1}{2}}\psi, A^{\frac{1}{2}}\psi) = |A^{\frac{1}{2}}\psi|^2 = |\psi|^2_{\frac{1}{2}}.$$

On the other hand, we have

$$\begin{aligned}(A\psi,\psi) &= -\sum_{i,j=1}^{n}\int_\Omega \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\psi(x)\right)\psi(x)dx,\\ &= \sum_{i,j=1}^{n}\int_\Omega a_{ij}(x)\frac{\partial}{\partial x_j}\psi(x)\frac{\partial}{\partial x_i}\psi(x)dx. \qquad (6.2)\end{aligned}$$

Using the ellipticity condition, we obtain

$$(A\psi,\psi) \ge \nu\sum_{i=1}^{n}\int_\Omega\left(\frac{\partial}{\partial x_i}\psi(x)\right)^2 dx = \nu|\nabla\psi|^2.$$

Consequently,

$$\sqrt{\nu}|\nabla\psi| \le |\psi|_{\frac{1}{2}}.$$

From (6.2), we obtain

$$(A\psi,\psi) \le \max_{1\le i,j\le n}|a_{i,j}|_{L^\infty}\sum_{i,j=1}^{n}\int_\Omega |\frac{\partial}{\partial x_j}\psi(x)||\frac{\partial}{\partial x_i}\psi(x)|dx.$$

By Young's inequality, we obtain

$$\begin{aligned}(A\psi,\psi) &\le \max_{1\le i,j\le n}|a_{i,j}|_{L^\infty}\frac{1}{2}\sum_{i,j=1}^{n}\int_\Omega\left(\left|\frac{\partial}{\partial x_i}\psi(x)\right|^2+\left|\frac{\partial}{\partial x_j}\psi(x)\right|^2\right)dx,\\ &\le \max_{1\le i,j\le n}|a_{i,j}|_{L^\infty}\sum_{i=1}^{n}\int_\Omega\left|\frac{\partial}{\partial x_i}\psi(x)\right|^2 dx,\\ &\le \max_{1\le i,j\le n}|a_{i,j}|_{L^\infty}|\nabla\psi|^2.\end{aligned}$$

Consequently,

$$|\psi|_{\frac{1}{2}} \le \sqrt{\max_{1\le i,j\le n}|a_{i,j}|_{L^\infty}}\ |\nabla\psi|.\ \square$$

The following Lemmas are needed to show the well posedness of Equation (6.1).

Lemma 6.3. *Let $\varphi \in \mathcal{C}_{\frac{1}{2}}$. Then*

$$\varphi(.)(x) \in L^2([-r,0]) \quad \text{for a.e. } x \in \Omega.$$

Proof. Let $\varphi \in \mathcal{C}_{\frac{1}{2}}$. Then $\varphi \in \mathcal{C}([-r, 0]; X)$, which implies that

$$\sup_{\theta \in [-r,0]} \int_{\Omega} |\varphi(\theta)(x)|^2 \, dx < \infty,$$

and

$$\int_{-r}^{0} \int_{\Omega} \left|\varphi(\theta)(x)^2\right| dx d\theta < \infty.$$

By Fubini's Theorem, we get

$$\int_{\Omega} \left(\int_{-r}^{0} |\varphi(\theta)(x)|^2 \, d\theta \right) dx < \infty.$$

Consequently,

$$\int_{-r}^{0} |\varphi(\theta)(x)|^2 \, d\theta < \infty \text{ for a.e. } x \in \Omega. \ \square$$

Let us introduce the linear operator $L : \mathcal{C}_{\frac{1}{2}} \to X$ defined for $\phi \in \mathcal{C}_{\frac{1}{2}}$ by

$$L(\phi) = -a_0 \phi(0) + \varepsilon \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \phi(-r_i) + \int_{-r_{n+1}}^{0} \beta(\theta) \phi(\theta) d\theta.$$

Proposition 6.4. *L is a bounded linear operator from $\mathcal{C}_{\frac{1}{2}}$ to X.*

Proof. L is decomposed as follows

$$L = L_1 + L_2,$$

where L_1 and L_2 are defined for $\varphi \in \mathcal{C}_{\frac{1}{2}}$ by

$$\begin{cases} L_1(\varphi) = -a_0 \varphi(0) + \varepsilon \sum_{i=1}^{n} \dfrac{\partial}{\partial x_i} \varphi(-r_i), \\ L_2(\varphi) = \displaystyle\int_{-r_{n+1}}^{0} \beta(\theta) \varphi(\theta) d\theta. \end{cases}$$

It follows in L^2-norm that

$$|L_1(\varphi)| \leq |a_0| \, |\varphi(0)| + \varepsilon \sum_{i=1}^{n} \left| \frac{\partial}{\partial x_i} \varphi(-r_i) \right|.$$

On the other hand, by Lemma 6.2 we have for $i = 1, ..., n$,

$$\left| \frac{\partial}{\partial x_i} \varphi(-r_i) \right| \leq |\nabla \varphi(-r_i)| \leq \frac{1}{\sqrt{\nu}} |\varphi(-r_i)|_{\frac{1}{2}} \leq \frac{1}{\sqrt{\nu}} |\varphi|_{\mathcal{C}_{\frac{1}{2}}}.$$

Therefore,

$$|\varphi(0)| \leq \left| A^{-\frac{1}{2}} \right| |\varphi(0)|_{\frac{1}{2}}.$$

Consequently, we deduce that L_1 is a bounded linear operator from $\mathcal{C}_{\frac{1}{2}}$ to X. Lemma 6.3 implies that $\varphi(.)(x) \in L^2(-r, 0)$ and

$$|L_2(\varphi)(x)|^2 \leq \int_{-r_{n+1}}^{0} \beta(\theta)^2 d\theta \int_{-r_{n+1}}^{0} |\varphi(\theta)(x)|^2 \, d\theta.$$

Then

$$\int_\Omega |L_2(\varphi)(x)|^2\,dx \le \int_{-r_{n+1}}^0 |\beta(\theta)|^2\,d\theta \int_{-r_{n+1}}^0 \left(\int_\Omega |\varphi(\theta)(x)|^2\,dx\right) d\theta$$

and

$$|L_2(\varphi)|^2 \le \int_{-r_{n+1}}^0 |\beta(\theta)|^2\,d\theta \int_{-r_{n+1}}^0 |\varphi(\theta)|^2 d\theta.$$

It follows that

$$|L_2(\varphi)|^2 \le \left|A^{-\frac{1}{2}}\right|^2 \int_{-r_{n+1}}^0 |\beta(\theta)|^2\,d\theta \int_{-r_{n+1}}^0 |A^{\frac{1}{2}}\varphi(\theta)|^2 d\theta$$

and

$$|L_2(\varphi)|^2 \le \left|A^{-\frac{1}{2}}\right|^2 r_{n+1} \left(\int_{-r_{n+1}}^0 |\beta(\theta)|^2\,d\theta\right) |\varphi|^2_{\mathcal{C}_{\frac{1}{2}}}.$$

Consequently, L_2 is a bounded linear operator from $\mathcal{C}_{\frac{1}{2}}$ to X. □

Let $f : \mathbb{R} \longrightarrow X$ be defined by

$$f(t)(x) = \Theta(t,x) \text{ for } t \in \mathbb{R} \text{ and } x \in \Omega,$$

and

$$\begin{cases} u(t)(x) = v(t,x) \text{ for } t \ge \sigma \text{ and } x \in \Omega, \\ \varphi(\theta)(x) = \varphi_0(\theta,x) \text{ for } \theta \le 0 \text{ and } x \in \Omega. \end{cases}$$

Then Equation (6.1) takes the following abstract form

$$\begin{cases} \dfrac{d}{dt}u(t) = -Au(t) + L(u_t) + f(t) \quad \text{for } t \ge \sigma, \\ u_\sigma = \varphi \in \mathcal{C}_{\frac{1}{2}} := \mathcal{C}([-r,0]\,; X_{\frac{1}{2}}). \end{cases} \tag{6.3}$$

Let $\varphi \in \mathcal{C}_{\frac{1}{2}}$. By Theorem 2.4, there exists a unique solution u of Equation (6.3) on $[\sigma - r, +\infty)$.

Let $(\mathcal{U}(t))_{t\ge 0}$ be the solution semigroup on $\mathcal{C}_{\frac{1}{2}}$ of Equation (6.3) and $\mathcal{A}_{\mathcal{U}}$ be its infinitesimal generator (see Preposition 2.5 and Proposition 2.6). In order to obtain the hyperbolicity of $(\mathcal{U}(t))_{t\ge 0}$, we add the following assumption

$(\mathbf{A_1})$ $\displaystyle\int_{-r_{n+1}}^0 |\beta(\theta)|\,d\theta < a_0.$

The next important result is needed to prove the hyperbolicity of $(\mathcal{U}(t))_{t\ge 0}$.

Lemma 6.5. *There exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon < \varepsilon_0$ *and for* $\lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) \ge 0$, *we have*

$$\{\mu \in \mathbb{C} : \mathrm{Re}(\mu) \ge 0\} \subset \rho\left(-A + \varepsilon \sum_{i=1}^n e^{-\lambda r_i} \frac{\partial}{\partial x_i}\right),$$

where $\rho\left(-A + \varepsilon \sum_{i=1}^n e^{-\lambda r_i} \dfrac{\partial}{\partial x_i}\right)$ *denotes the resolvent set of the differential operator*

$$-A + \varepsilon \sum_{i=1}^n e^{-\lambda r_i} \frac{\partial}{\partial x_i}.$$

Proof. Let $\lambda, \mu \in \mathbb{C}$ be such that $\mathrm{Re}(\lambda), \mathrm{Re}(\mu) \geq 0$. Then $\mu \in \rho(-A)$ and

$$\mu I + A - \varepsilon \sum_{i=1}^{n} e^{-\lambda r_i} \frac{\partial}{\partial x_i} = \left(I - \varepsilon \sum_{i=1}^{n} e^{-\lambda r_i} \frac{\partial}{\partial x_i} R(\mu, -A) \right) (\mu I + A).$$

For $i = 1, ..., n$, we have in L^2-norm

$$\left| e^{-\lambda r_i} \frac{\partial}{\partial x_i} R(\mu, -A) \right| \leq \left| \nabla R(\mu, -A) \right|.$$

Lemma 6.2 implies that for $q \in L^2(\Omega)$

$$\left| \nabla R(\mu, -A) q \right| \leq \frac{1}{\sqrt{\nu}} \left| R(\mu, -A) q \right|_{\frac{1}{2}}.$$

On the other hand,

$$A^{\frac{1}{2}} R(\mu, -A) q = \int_0^{+\infty} e^{-\mu t} A^{\frac{1}{2}} T(t) q dt.$$

From (**v**) of Theorem 2.2, we obtain that

$$\left| R(\mu, -A) \right|_{\frac{1}{2}} \leq M_{\frac{1}{2}} \int_0^{+\infty} \frac{e^{(\lambda_0 - \mathrm{Re}\,\mu) t}}{t^{\frac{1}{2}}} dt$$

and

$$\left| R(\mu, -A) \right|_{\frac{1}{2}} \leq M_{\frac{1}{2}} \int_0^{+\infty} \frac{e^{\lambda_0 t}}{t^{\frac{1}{2}}} dt.$$

Using formula (3.2), we get

$$\left| \nabla R(\mu, -A) \right| \leq M_{\frac{1}{2}} \frac{\sqrt{\pi}}{\sqrt{-\lambda_0 \nu}}.$$

It follows that

$$\left| \varepsilon \sum_{i=1}^{n} e^{-\lambda r_i} \frac{\partial}{\partial x_i} R(\mu, -A) \right| \leq \varepsilon n M_{\frac{1}{2}} \frac{\sqrt{\pi}}{\sqrt{-\lambda_0 \nu}}.$$

Let $\varepsilon_0 = \dfrac{1}{n M_{\frac{1}{2}}} \sqrt{\dfrac{-\lambda_0 \nu}{\pi}}$. Then for $\varepsilon < \varepsilon_0$, we have

$$\left| \varepsilon \sum_{i=1}^{n} e^{-\lambda r_i} \frac{\partial}{\partial x_i} R(\mu, -A) \right| < 1.$$

Hence the operator

$$I - \varepsilon \sum_{i=1}^{n} e^{-\lambda r_i} \frac{\partial}{\partial x_i} R(\mu, -A)$$

is invertible. This implies that for $\varepsilon < \varepsilon_0$.

$$\mu \in \rho \left(-A + \varepsilon \sum_{i=1}^{n} e^{-\lambda r_i} \frac{\partial}{\partial x_i} \right). \ \square$$

Proposition 6.6. *Let $\varepsilon < \varepsilon_0$. Then, $\sigma(\mathcal{A}_{\mathcal{U}}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$.*

Proof. Let $\lambda \in \sigma(\mathcal{A}_{\mathcal{U}})$, then there exists $y \in D(A)$, $y \neq 0$ such that $\Delta(\lambda)y = 0$, where $\Delta(\lambda)$ is given by

$$\Delta(\lambda)y = \lambda y + Ay + a_0 y - \varepsilon \sum_{i=1}^{n} e^{-\lambda r_i} \frac{\partial}{\partial x_i} y - \left(\int_{-r_{n+1}}^{0} \beta(\theta) e^{\lambda\theta} d\theta \right) y, \text{ for } y \in D(A)\backslash\{0\},$$

which implies that

$$\lambda y + Ay + a_0 y - \varepsilon \sum_{i=1}^{n} e^{-\lambda r_i} \frac{\partial}{\partial x_i} y - \left(\int_{-r_{n+1}}^{0} \beta(\theta) e^{\lambda\theta} d\theta \right) y = 0$$

and

$$\lambda + a_0 - \int_{-r_{n+1}}^{0} \beta(\theta) e^{\lambda\theta} d\theta \in \sigma_p \left(-A + \varepsilon \sum_{i=1}^{n} e^{-\lambda r_i} \frac{\partial}{\partial x_i} \right).$$

We proceed by contradiction and assume that there exits a characteristic value λ such that $\operatorname{Re}\lambda \geq 0$. By assumption $(\mathbf{A_1})$, we can see that

$$\operatorname{Re}(\lambda) + a_0 - \int_{-r_{n+1}}^{0} \beta(\theta) e^{\operatorname{Re}(\lambda)\theta} \cos(\operatorname{Im}(\lambda)\theta) d\theta \geq 0.$$

Lemma 6.5 implies that

$$\lambda + a_0 - \int_{-r_{n+1}}^{0} \beta(\theta) e^{\lambda\theta} d\theta \in \rho \left(-A + \varepsilon \sum_{i=1}^{n} e^{-\lambda r_i} \frac{\partial}{\partial x_i} \right).$$

This gives a contradiction. Consequently,

$$\sigma(\mathcal{A}_{\mathcal{U}}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) < 0\}. \ \square$$

Which implies that the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic.

In the next, we suppose that $\Theta : \mathbb{R} \times \Omega \to \mathbb{R}$ is continuous, in order to obtain a unique bounded solution on $\mathbb{R}$ of the following equation

$$\frac{d}{dt} u(t) = -Au(t) + L(u_t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{6.4}$$

we assume that

$(\mathbf{A_2})$ For every $t \in \mathbb{R}$, the function $\Theta(t, .) \in L^2(\Omega)$ and the function $f : t \to \Theta(t, .)$ from $\mathbb{R}$ to $L^2(\Omega)$ is continuous and bounded, namely,

$$\sup_{t\in\mathbb{R}} \left(\int_{\Omega} |\Theta(t,x)|^2 \, dx \right) < +\infty.$$

Corollary 6.7. *Let $\varepsilon < \varepsilon_0$. Then Equation (6.4) has a unique bounded solution on $\mathbb{R}$.*

Proof. By Theorem 5.9, Proposition 6.6 and Assumption $(\mathbf{A_2})$, we conclude that Equation (6.4) has a unique bounded solution on $\mathbb{R}$. $\square$

Thanks to Theorem 5.9, we obtain the following result.

Corollary 6.8. *If the function f is almost automorphic (resp. almost periodic, resp. periodic), then the only bounded solution of Equation (6.4) is almost automorphic (resp. almost periodic, resp. periodic).*

Example. (i) Let

$$\Theta(t,x) = (\sin(\alpha t) + \sin(\beta t))\,\Theta_0(x),$$

where $\dfrac{\alpha}{\beta} \notin \mathbb{Q}$ and $\Theta_0 : \Omega \to \mathbb{R}$ is in $L^2(\Omega)$. We deduce that the function $f : \mathbb{R} \to L^2(\Omega)$ is almost periodic.

(ii) As an example of almost automorphic function, we propose

$$\Theta(t,x) = \sin\left(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\right)\Theta_0(x),$$

where Θ_0 is the same function as in **(i)**.

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