Inverse problem of Fractional calculus of variations for Partial differential equations

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Abstract

The current paper aims at finding out a Lagrangian structure for some partial differential equations including the Stokes equations, the fractional wave equation, the diffusion or fractional diffusion equations, using the fractional embedding theory of continuous Lagrangian systems.

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1. Introduction

This paper is an introduction to the framework of embedding theories of Lagrangian systems initiated in [6]. We review results about applications of the fractional embedding procedure [7] to partial differential equations.

Many classical partial differential equations possess a fractional analogue, like the wave equation or the diffusion equation. In general, these fractional analogue are obtained by changing the classical time derivative by a fractional one, which can be Riemann-Liouville, Caputo or another one. Of course, these generalizations are supported by physical arguments but these

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equations always seem to be pure formal extensions. In this paper, we have choosen to follow a different strategy.

For example, the fractional wave equation comes from the wave equation which possesses a (continuous) Lagrangian structure. We impose to the fractional analogue to possess a fractional analogue of this Lagrangian structure. This can be done using the framework of the fractional embedding of Lagrangian systems. In that case, the fractional PDE is also solution of a variational principle which is a fractional deformation of the classical one.

Using this strategy we can also discuss the inverse problem of fractional calculus of variations for classical partial differential equations, like the diffusion equation or Stokes equations, in order to obtain a Lagrangian representation of such PDEs.

The interest of such results is at least twofold:

- First of all, we now have an intrinsic object, the Lagrangian, which controls the dynamical behaviour of the PDE.

- Secondly, this intrinsic structure can be used to derive more adapted numerical schemes for these equations. This will be discussed in a forthcoming paper.

Section 2 gives a general introduction to the strategy of embedding theories. In section 3 we describe the fractional operators which are used in this paper and some of their properties. In section 4 we give a self-contain introduction to the fractional embedding theory of Lagrangian system developped in [7]. In section 5, we derive explicit Lagrangian densities for the fractional wave equation, the fractional diffusion equation and the Stokes equation. We discuss open problems in section 6.

2. Embedding

This section is an introduction to the idea of embedding for differential operators as developped for example in ([6],[7],[4]).

2.1. Formal presentation

We refer to [7] where the notion of embedding is defined in general as well as the notion of deformation.

An embedding is the data of two ingredients:

- A set A

- An operator D acting on A

Using A and D we can associated to a differential operator $P[d/dt] = \sum_i a_i \frac{d^i}{dt^i}$ an embedded analogue

$$\mathrm{Emb}(P[d/dt]) = \sum_i a_i D^i,$$

acting on A.

2.2. An example: Schwartz's embedding

In this case the set A is $\mathcal{D}'(\Omega)$ the set of distributions on Ω. We have a natural map from C^0 into $\mathcal{D}'(\Omega)$ defined on $\mathcal{D}(\Omega)$ by

$$\iota(f) : \phi \to \int_\Omega f\phi$$

The classical derivative d/dt has an analogue on $\mathcal{D}'(\Omega)$ which we denote by $D : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ which for $T \in \mathcal{D}'(\Omega)$ is defined by

$$D(T)(\phi) = -T(\phi')$$

We remark that we have

$$\iota(f') = D(\iota(f))$$

Then we associate to each differential operator $\sum a_i \frac{d^i}{dt^i}$ an operator acting on $\mathcal{D}'(\Omega)$

$$\sum a_i D^i$$

2.3. Problems induced by embedding of equations

2.3.1. Embedding and change of variables

The form of a given differential operator representing an equation is not an intrinsic object. It depends mainly on the coordinates system which is used to described the dynamics. In the classical case, a change of variables, i.e. a map

$$y = h(x)$$

with h a bijective map of class C^1, deos not affect the set of solutions as we have a conjugacy between the solution $\phi(x)$ in the variable x and $\psi(y)$ in the variable y by

$$\psi \circ h = h^{-1} \circ \phi$$

However, this simple action has not always a good behaviour with respect to embedding. Indeed, if we denote by O the differential operator asoociated to the equation in the set of variable x and by $\tilde{O}$ the associated differential operator in the set of variable y, we have not in general a simple link between the embedded differential operators or solutions of the embedded equations. This is mainly due to the fact that an embedding does not respect in general the classical rule of derivation and as a consequence under a change of variables we do not obtain the same thing.

2.3.2. Embedding and generalized solutions

An embedding is usually used to extend a given equation in order to obtain a more general set of solutions as for example when one uses the Schwartz's embedding for partial differential equations. However, doing this, we have in general a problem: the set of allowed solutions is too big. As an example, when one uses Schwartz's distributions to compute the solutions of the partial differential equation $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2} = 0$ we obtain (see [17],p.4-5), for ditributions which are functions, $u(x,y) = f(x-y) + g(x-y)$ where f and g need not to posess second order derivatives.

As a consequence, one must introduce a selection criterion in order to obtain relevant or "physical" solutions, as for example the classical entropic conditions of Peter Lax. The same phenomenon exists in all embedding theory.

2.4. Embedding and intrinsic structures

We have then an infinity of possibilities for the embed equations depending on the choice of the underlying coordinates system used to describe the dynamics. A way to overcome the previous problems is to focus on intrinsic structures for equations.

2.4.1. Intrinsic structures: the case of Lagrangian systems

Most of partial or ordinary differential equations which are studied have a physical origine. Some of them are not only phenomenological equations but are derived via first principles of physics. This is the case for ordinary differential equations which comes in classical mechanics. These equations can be obtain via a variational principle, the least action principle, which says that solutions of these equations correspond to extremals of a functional called a Lagrangian functional. The least action principle is independant of the coordinates system. A Lagrangian being given the form of the equation is deduced from the functional. An idea in order to prescribe the form of the embed equation is then to look for the behaviour of the variational principle under the embedding.

2.4.2. The coherence problem

A lagrangian functional is a functional of the form

$$\mathcal{L}(x) = \int_a^b L(x(t), \dot{x}(t), t)\, dt,$$

where L is a function, and $x : t \in [a, b] \to \mathbb{R}^n$, $n \geq 1$ is C^1. Under embedding we can usually give a meaning to these functional. We denote by $\mathcal{L}_{\text{emb}}(X)$, $X \in A$ the new functional. Assuming that one can developp a calculus of variations for these new kind of functionals, we obtain embed analogue of the Euler-Lagrange equation that we denote $EAEL$ in the following. This programm has been done in the non-differentiable [4], stochastic [6] and fractional case [7].

We have now the following diagramm:

$$\begin{array}{ccl} \mathcal{L}(x) & \stackrel{\text{Emb}}{\longrightarrow} & \mathcal{L}_{\text{emb}}(X) \\ & & \downarrow \\ & & \text{Embedded analogue of Euler-Lagrange equation} \end{array}$$

This diagramm must be completed by the following one

$$\begin{array}{ccc} \mathcal{L}(x) & & \\ \downarrow & & \\ \text{Euler-Lagrange equation} & \xrightarrow{\text{Emb}} & \text{Embedded Euler-Lagrange equation} \end{array}$$

As a consequence, we have two possible generalizations of the classical Euler-Lagrange equation. One by embedding and the second one by the embedded functional. The second one keeps the fundamental structure of the equation and the first one is natural. One condition that we can imposed is the following property for the embedding scheme:

Coherence: *An embedding is coherent if* $EAEL = \mathrm{Emb}(EL)$.

An embedding is not always coherent (see [6],[7]) and a non trivial problem is to find conditions under which an embedding can be made coherent.

3. Reminder about fractional calculus

3.1. Left and right Riemann-Liouville derivatives

We define the left and right Riemann-Liouville derivatives following [12, 15, 14, 11]. Let $a, b \in \mathbb{R}$, $a < b$ and $\alpha \in \mathbb{R}$.

Definition 1 (Left Riemann-Liouville Fractional integral). *Let x be a measurable function defined on (a, b), and $\alpha > 0$. Then the left Riemann-Liouville fractional integral of order α is defined to be, when it exists,*

$$ {}_a\mathrm{D}_t^{-\alpha} X_t := \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) ds. \tag{1}$$

Definition 2 (Right Riemann-Liouville Fractional integral). *Let x be a measurable function defined on (a, b), and $\alpha > 0$. Then the right Riemann-Liouville fractional integral of order α is defined to be, when it exists,*

$$ {}_t\mathrm{D}_b^{-\alpha} X_t := \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} x(s) ds. \tag{2}$$

We use the abbreviation RL for Riemann-Liouville. Left and right (RL) integrals satisfy some important properties like the semi-group property. We refer to [15] for more details.

Definition 3 (Left and right RL fractional derivative). *Let $\alpha > 0$, the left and right Riemann-Liouville derivative of order α, denoted by ${}_a\mathrm{D}_t^\alpha$ and ${}_t\mathrm{D}_b^\alpha$ respectively, are defined by*

$$ {}_a\mathrm{D}_t^\alpha X_t = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} x(s) ds, \tag{3} $$

and

$$ {}_t\mathrm{D}_b^\alpha X_t = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (t-s)^{n-\alpha-1} x(s) ds, \tag{4} $$

where $n \in \mathbb{N}$ is such that $n-1 \leq \alpha < n$.

If $\alpha = m$, $m \in \mathbb{N}^*$, we denote by $C^m(]a,b[)$ the set of mappings having m continuous derivatives on $]a,b[$. For $x \in C^m(]a,b[)$ we have

$$ {}_a\mathrm{D}_t^m x = \frac{d^m x}{dt^m}, \quad {}_t\mathrm{D}_b^m = -\frac{d^m x}{dt^m}. \tag{5} $$

This last relation which ensures that the left and right Riemann-Liouville (RL) derivatives coincide with the classical derivative for positive integer will be of fundamental importance in what follows.

If $x \in C^0(]a,b[)$ with left and right-derivatives at point t denoted by $\dfrac{d^+x}{dt}$ and $\dfrac{d^-x}{dt}$ respectively then

$$ {}_a\mathrm{D}_t^m x = \frac{d^+ x}{dt}, \quad {}_t\mathrm{D}_b^m = -\frac{d^- x}{dt}. \tag{6} $$

In what follows, we denote by ${}_a^\alpha\mathbf{E}$, $\mathbf{E}_b^\alpha$ and ${}_a\mathbf{E}_b(\alpha)$ the functional spaces defined by

$$ \begin{array}{l} {}_a^\alpha\mathbf{E} = \{x \in C([a,b]),\ {}_a\mathrm{D}_t^\alpha x \text{ exists}\}, \\ \mathbf{E}_b^\alpha = \{x \in C([a,b]),\ {}_t\mathrm{D}_b^\alpha x \text{ exists}\}, \end{array} \tag{7} $$

and

$$ {}_a\mathbf{E}_b(\alpha) = {}_a^\alpha\mathbf{E} \cap \mathbf{E}_b^\alpha. \tag{8} $$

Remark 1. *Of course the set* ${}_a\mathbf{E}_b(\alpha)$ *is non-empty. Following [[15] Lemma 2.2 p.35] we have* $AC([a,b]) \subset {}_a\mathbf{E}_b(\alpha)$*, where* $AC([a,b])$ *is the set of absolutely continuous functions on the interval* $[a,b]$ *[see [15] Definition 1.2].*

The operators of ordinary differentiation of integer order satisfy a commutativity property and the law of exponents (the semi-group property) *i.e.*

$$\frac{d^n}{dt^n} \circ \frac{d^m}{dt^m} = \frac{d^m}{dt^m} \circ \frac{d^n}{dt^n} = \frac{d^{n+m}}{dt^{n+m}}. \tag{9}$$

These two properties in general fail to be satisfied by the left and right fractional RL derivatives. We refer to ([11] §.IV.6) and [[10] p.233] for more details and examples. These bad properties are responsible for several difficulties in the study of fractional differential equations. We refer to [14] for more details.

3.2. Left and right fractional derivatives

In some cases, we need that our fractional operators satisfy additional properties like the *semi-group property*. Following [8] we introduce the *left and right fractional derivatives* as well as convenient functional spaces on which we have the semi-group property.

Definition 4 (Left fractional derivative). *Let* x *be a function defined on* $\mathbb{R}$*,* $\alpha > 0$*,* n *be the smallest integer greater than* α $(n-1 \leq \alpha < n)$*, and* $\sigma = n - \alpha$*. Then the left fractional derivative of order* α *is defined to be*

$$\begin{aligned} \mathbb{D}^\alpha x(t) &:= {}_\infty D_t^\alpha x(t) = \frac{d^n}{dt^n} {}_\infty D_t^{-\alpha} x(t) \\ &= \frac{1}{\Gamma(\sigma)} \frac{d^n}{dt^n} \int_{-\infty}^t (t-s)^{\sigma-1} x(s) ds. \end{aligned} \tag{10}$$

Definition 5 (Right fractional derivative). *Let* x *be a function defined on* $\mathbb{R}$*,* $\alpha > 0$*,* n *be the smallest integer greater than* α $(n-1 \leq \alpha < n)$*, and* $\sigma = n - \alpha$*. Then the right fractional derivative of order* α *is defined to be*

$$\begin{aligned} \mathbb{D}_*^\alpha x(t) &:= {}_t D_\infty^\alpha x(t) = (-1)^n \frac{d^n}{dt^n} {}_t D_\infty^{-\alpha} x(t) \\ &= \frac{(-1)^n}{\Gamma(\sigma)} \frac{d^n}{dt^n} \int_t^{\infty} (s-t)^{\sigma-1} x(s) ds. \end{aligned} \tag{11}$$

If $\overline{\mathrm{Supp}(x)} \subset (a,b)$ we have $\mathbb{D}^\alpha x = {}_a\mathrm{D}_t^\alpha x$ and $\mathbb{D}_*^\alpha x = {}_t\mathrm{D}_b^\alpha x$.

In [8] several useful functional spaces are introduced. Let $I \subset \mathbb{R}$ be an open interval (which may be unbounded). We denote by $C^\infty(I)$ the set of infinitely differentiable mappings and by $C_0^\infty(I)$ the set of all functions $x \in C^\infty(I)$ that vanish outside a compact subset K of I.

Definition 6 (Left fractional derivative space). *Let $\alpha > 0$. Define the semi-norm*

$$| x |_{J_L^\alpha(\mathbb{R})} := \| \mathbb{D}^\alpha x \|_{L^2(\mathbb{R})}, \tag{12}$$

and norm

$$\| x \|_{J_L^\alpha(\mathbb{R})} := \left(\| x \|_{L^2(\mathbb{R})}^2 + | x |_{J_L^\alpha(\mathbb{R})}^2 \right)^{1/2}. \tag{13}$$

and let $J_L^\alpha(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ with respect to $\| \cdot \|_{J_L^\alpha(\mathbb{R})}$.

Similarly, we can defined the right fractional derivative space.

Definition 7 (Right fractional derivative space). *Let $\alpha > 0$. Define the semi-norm*

$$| x |_{J_R^\alpha(\mathbb{R})} := \| \mathbb{D}_*^\alpha x \|_{L^2(\mathbb{R})}, \tag{14}$$

and norm

$$\| x \|_{J_R^\alpha(\mathbb{R})} := \left(\| x \|_{L^2(\mathbb{R})}^2 + | x |_{J_R^\alpha(\mathbb{R})}^2 \right)^{1/2}. \tag{15}$$

and let $J_R^\alpha(\mathbb{R})$ denote the closure of $C_0^\infty(\mathbb{R})$ with respect to $\| \cdot \|_{J_R^\alpha(\mathbb{R})}$.

We now assume that I is a *bounded* open subinterval of $\mathbb{R}$. We restrict the fractional derivative spaces to I.

Definition 8. *Define the spaces $J_{L,0}^\alpha(I)$, $J_{R,0}^\alpha(I)$ as the closure of $C_0^\infty(I)$ under their respective norms.*

These spaces have very interesting properties with respect to $\mathbb{D}$ and $\mathbb{D}_*$. In particular, we have the following semi-group property:

Lemma 1. *For $x \in J_{L,0}^\beta(I)$, $0 < \alpha < \beta$ we have*

$$\mathbb{D}^\beta x = \mathbb{D}^\alpha \mathbb{D}^{\beta-\alpha} x \tag{16}$$

and similarly for $x \in J_{R,0}^\beta(I)$,

$$\mathbb{D}_*^\beta x = \mathbb{D}_*^\alpha \mathbb{D}_*^{\beta-\alpha} x. \tag{17}$$

We refer to [[8] Lemma 2.9] for a proof. In particular, choosing $\beta = 2\alpha$, $\alpha > 0$, $x \in J^{2\alpha}_{L,0}(I)$, we have $\mathbb{D}^{2\alpha}x = \mathbb{D}^{\alpha}\mathbb{D}^{\alpha}x$ and for $x \in J^{2\alpha}_{R,0}(I)$ we obtain $\mathbb{D}^{2\alpha}_{*}x = \mathbb{D}^{\alpha}_{*}\mathbb{D}^{\alpha}_{*}x$.

The fractional derivative spaces $J^{\alpha}_{L,0}(I)$ and $J^{\alpha}_{R,0}(I)$ have been characterized when $\alpha > 0$. We denote by $H^{\alpha}_{0}(I)$ the *fractional Sobolev space.*

Theorem 1. *Let $\alpha > 0$. Then the $J^{\alpha}_{L,0}(I)$, $J^{\alpha}_{R,0}(I)$ and $H^{\alpha}_{0}(I)$ spaces are equal.*

We refer to [[8] Theorem 2.13] for a proof. In fact, when $\alpha \neq n - 1/2$, $n \in \mathbb{N}$ we have a stronger result as the $J^{\alpha}_{L,0}(I)$, $J^{\alpha}_{R,0}(I)$ and $H^{\alpha}_{0}(I)$ spaces have equivalent semi-norms and norms.

We now introduce the fractional operator that we need in the following.

Definition 9. *We denote by d^{α}_{μ} the operator defined by*

$$d^{\alpha}_{\mu} = \frac{\mathbb{D}^{\alpha} + \mathbb{D}^{\alpha}_{*}}{2} + i\mu\frac{\mathbb{D}^{\alpha} - \mathbb{D}^{\alpha}_{*}}{2}, \tag{18}$$

for $\mu = 1, 0, \pm i$.

4. Reminder about the fractional embedding of continuous Lagrangian systems

We follow [7]. We refer to [18] and [2] for related works on the fractional calculus of variations for fields.

Let $d \in \mathbb{N}$. We consider a Lagrangian function Ł defined on $\mathbb{R}\times\mathbb{R}^{d}\times\mathbb{R}\times\mathbb{C}\times\mathbb{R}^{d}$ and denoted by

$$Ł(t, x_1, \ldots, x_d, y, v, w_1, \ldots, w_d). \tag{19}$$

In what follows, we use the terminology of *Lagrangian density* for a function Ł of the form (19). We denote x for $(x_1, \ldots, x_d)$. A Lagrangian density is *admissible* if $L(t, x, y, w)$ is holomorphic with respect to v, differentiable with respect to x and w, and real when $v \in \mathbb{R}$.

Let $\Re$ be a fixed region of $\mathbb{R}^d$ and $a < b$, $a, b \in \mathbb{R}$. We consider the functional

$$\mathcal{L}_{a,b,\Re}(u) = \int_a^b \int_{\Re} \text{Ł}(t, x, u(t,x), \partial_t u(t,x), \partial_x u(t,x)) \, dx \, dt, \tag{20}$$

acting on a function $u : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ which is usually called a *field*, which is of class C^1 in all its variables and where

$$\partial_x u(t,x) = (\partial_{x_1} u(t,x), \ldots, \partial_{x_d} u(t,x)). \tag{21}$$

A *variation* for a field $u(t,x)$ is defined as a function of the form

$$u_\epsilon(t,x) = u(t,x) + \epsilon h(t,x), \tag{22}$$

where $0 < \epsilon << 1$ is a small parameter and with h satisfying the *boundaries conditions*

$$h(a,x) = h(b,x) = 0 \quad \text{and} \quad h(t, \partial\Re) = 0, \tag{23}$$

where $\partial\Re$ denotes the boundary of $\Re$.

Using the fractional embedding procedure, we look for the following class of *fractional densities*:

Definition 10. *Let Ł be an admissible Lagrangian density. The fractional functional associated to Ł is defined by*

$$\mathcal{L}^\alpha_{a,b}(u) = \int_a^b \int_{\Re} \text{Ł}(t, x, u(t,x), d^\alpha_\mu u(t,x), \partial_x u(t,x)) \, dx \, dt, \tag{24}$$

for fields $u(t,x) \in {}^\alpha_a\mathbb{F}_b(\Re)$*, the set of fields smooth with respect to* x *and in* ${}_a\mathbf{E}_b(\alpha)$ *with respect to* t*.*

We consider two spaces of variations for fields. For a field $h : (t,x) \mapsto h(t,x)$ we denote by h_t and h_x the partial maps $h_t : x \mapsto h(t,x)$ where t is fixed and $h_x : t \mapsto h(t,x)$ where x is fixed.

Definition 11 (Spaces of variations for fields). *We denote by* $\mathbf{Var}^\alpha(a,b,\Re)$ *the set of fields satisfying*

$$\mathbf{Var}^\alpha(a,b,\Re) = \left\{ \begin{array}{c} h(t,x), h_t \in C^1, \ h_x \in {}_a\mathbf{E}_b(\alpha), \\ h(a,x) = h(b,x) = 0, \ h(t,\partial\Re) = 0 \end{array} \right\}, \tag{25}$$

and by $\mathbf{Var}_0^\alpha(a,b,\Re)$ *the set of fields defined by*

$$\mathbf{Var}_0^\alpha(a,b,\Re) = \left\{ \begin{array}{c} h(t,x), h_t \in C^1, \ h_x \in {}_a\mathbf{E}_b(\alpha), \\ h(a,x) = h(b,x) = 0, \\ h(t,\partial\Re) = 0, \ {}_a\mathrm{D}_t^\alpha h = {}_t\mathrm{D}_b^\alpha h \end{array} \right\}. \tag{26}$$

We denote by $\mathcal{P}$ either $\mathbf{Var}^\alpha(a,b\Re)$ or $\mathbf{Var}_0^\alpha(a,b,\Re)$. We have the following notion of differentiability for fractional functionals:

Definition 12. *Let* $Ł$ *be an admissible Lagrangian density and* $\mathcal{L}_{a,b}^\alpha$ *the associated fractional functional. The functional* $\mathcal{L}_{a,b}^\alpha$ *is called* $\mathcal{P}$*-differentiable at* u*, where* u *is a field, if*

$$\mathcal{L}_{a,b}^\alpha(u+\epsilon h) - \mathcal{L}_{a,b}^\alpha(u) = \epsilon d\mathcal{L}_{a,b}^\alpha(u,h) + o(\epsilon), \tag{27}$$

for all $h \in \mathcal{P}$*, where* $d\mathcal{L}_{a,b}^\alpha(u,h)$ *is a linear functional of* h*.*

As a consequence, we define the following notion of extremal:

Definition 13. *Let* L *be an adimissible density and* $\mathcal{L}^\alpha$ *the associated fractional functional. A* $\mathcal{P}$*-extremal for* $\mathcal{L}^\alpha$ *is a field* $u(x,t)$ *such that* $d\mathcal{L}^\alpha(u,h) = 0$ *for all* $h \in \mathcal{P}$*.*

The main result of [7] is:

Theorem 2 (Fractional least-action principle for fields). *Let* L *be an admissible Lagrangian density and* $\mathcal{L}_{a,b}^\alpha$ *the associated fractional functional. A necessary and sufficient condition for a field* u *to be a* $\mathbf{Var}^\alpha(a,b,\Re)$*-extremal is that it satisfies the fractional Euler-Lagrange equation for fields* $(FELF)_{-\mu}^\alpha$ *where*

$$\frac{\partial Ł}{\partial y}(z_\alpha(t,x)) - d_\mu^\alpha \left[\frac{\partial Ł}{\partial v}(z_\alpha(t,x)\right] - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left[\frac{\partial Ł}{\partial w_i}(z_\alpha(t,x))\right] = 0 \quad (FELF)_\mu^\alpha$$

where $z_\alpha(t,x) = (t,x,u(t,x), d_\mu^\alpha(t,x), \partial_x u(t,x))$*.*

The main problem with this theorem is that it does not provide a *coherent* embedding scheme of Lagrangian densities over fractional functional spaces. Indeed using the d_μ^α embedding of differential operators on the classical Euler-Lagrange equation we obtain $(FELF)_\mu^\alpha$ which is not the equation that we obtain using the fractional calculus of variation. Indeed, we obtain $(FELD)_{-\mu}^\alpha$ using theorem 2.

In order to bypass this problem, we use the set of real variations $\mathbf{Var}_0^\alpha(a,b,\Re)$.

Theorem 3 (Weak Fractional least-action principle for fields). *Let* Ł *be an admissible Lagrangian density and* $\mathcal{L}^{\alpha}_{a,b}$ *the associated fractional functional. A sufficient condition for a field* u *to be a* $\mathbf{Var}^{\alpha}_{0}(a,b,\Re)$*-extremal is that it satisfies the fractional Euler-Lagrange equation for fields* $(FELF)^{\alpha}_{\mu}$*.*

As a consequence, we have a coherent embedding but under a strong constraint on the set of variations.

5. Inverse problem of fractional calculus of variations for partial differential equations

5.1. The fractional wave equation

In this section, we derive the *fractional wave equation* defined in [16] as the extremals of a fractional continuous Lagrangian systems.

The equation describing waves propagating on a stretched string of constant linear mass density ρ under constant tension T is

$$\rho\frac{\partial^2 u(t,x)}{\partial t^2} = T\frac{\partial^2 u(t,x)}{\partial x^2}, \tag{28}$$

where $u(t,x)$ denotes the amplitude of the wave at position x along the string at time t. The wave equation corresponds to the extremals of the Lagrangian density

$$\text{Ł}(t,x,y,v,w) = \frac{\rho}{2}v^2 - \frac{T}{2}w^2. \tag{29}$$

In [16], the authors define the fractional analogue of the wave equation by changing the classical derivative by a fractional one. Using our notations, the definition of the fractional wave equation is:

Definition 14. *The fractional wave equation of order* $\alpha > 0$ *is the fractional differential equation*

$$-\rho\mathbb{D}^{2\alpha}u = T\frac{\partial^2 u}{\partial x^2}. \tag{30}$$

A natural demand with respect to this generalization which is just a formal manipulation on equations, is to keep a more structural property of the wave equation, namely the fact that it derives from a least-action principle. Using our fractional embedding procedure, we are able to explicit such a fractional Lagrangian framework for the fractional wave equation.

In the following we work with the fractional embedding associated to d^α_μ.

Theorem 4. *The d^α_μ-fractional embedding of the continuous Euler-Lagrange equation associated to (29) is given by*

$$-\rho d^\alpha_\mu \circ d^\alpha_\mu u = T\partial^2_{x^2} u. \tag{31}$$

We can specialized by choosing $\mu = -i$. In that case $d^\alpha_{-i} = \mathbb{D}^\alpha$ and satisfies a semi-group property (see lemma 1). As a consequence, we obtain:

Corollary 1. *The $\mathbb{D}^\alpha$-fractional embedding of the continuous Euler-Lagrange equation associated to (29) is given by*

$$-\rho \mathbb{D}^{2\alpha} u = T\partial^2_{x^2} u. \tag{32}$$

Moreover, using the weak coherence theorem, we have:

Theorem 5. *Solutions of the fractional wave equation (32) of order $\alpha > 0$ correspond to weak-extremals of the $\mathbb{D}^\alpha$-fractional functional associated to L.*

Up to the author knowledge, this is the first time that the fractional wave equation is derived via a fractional variational principle. In particular, the previous derivation has the advantage to keep the continuous Lagrangian structure underlying the classical wave equation.

5.2. The fractional diffusion equation

The *fractional diffusion equation* of order $0 < \alpha < 1$ is defined by

$$\mathbb{D}^\alpha u(t,x) = a^2 \frac{\partial^2 u(t,x)}{\partial x^2}. \tag{33}$$

It is defined by Wyss in [19]. For $\alpha = 1$ we recover the classical diffusion equation.

The aim of this section is to derive a fractional Lagrangian formulation of the fractional wave equation of order $0 < \alpha \leq 1$, then including the classical diffusion equation. In the contrary of the fractional wave equation, the diffusion equation is recovered thanks to a still fractional variational principle.

Let us consider the Lagrangian function Ł defined on $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{C} \times \mathbb{C}$ by

$$Ł(t, x, y, v, w) = \frac{1}{2} v^2 - \frac{a^2}{2} w^2, \tag{34}$$

where $\rho \in \mathbb{R}$, $a \in \mathbb{R}$.

Theorem 6. *The $d_\mu^{\alpha/2}$-fractional embedding of the continuous Euler-Lagrange equation associated to (34) is given by*

$$d_\mu^{\alpha/2} \circ d_\mu^{\alpha/2} u = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{35}$$

Choosing $\mu = -i$, we obtain $d_{-i}^\alpha = \mathbb{D}^\alpha$. As $\mathbb{D}^\alpha$ satisfies a semi-group property, we obtain:

Theorem 7. *The $\mathbb{D}^{\alpha/2}$-fractional embedding of the continuous Euler-Lagrange equation associated to (34) is given by*

$$\mathbb{D}^\alpha u = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{36}$$

It must be noted that even for $\alpha = 1$, the diffusion equation is recovered using a fractional embedding procedure, namely the $\mathbb{D}^{1/2}$-fractional embedding procedure.

The main result of this section is that this fractional embedding of the diffusion equation has an additional structure, a Lagrangian one.

Theorem 8. *Solutions of the fractional wave equation (33) of order* $0 < \alpha \leq 1$ *correspond to weak-extremals of the* $\mathbb{D}^{\alpha/2}$*-fractional functional associated to (34).*

This result seems new, even for the case $\alpha = 1$.

5.3. The incompressible Stokes equations

The incompressible Stokes equation looks like

$$\frac{\partial u}{\partial t} = \nu \Delta_x u - \nabla_x p. \tag{37}$$

We consider the following one parameter deformation of the incompressible Stokes equation:

$$\mathbb{D}_t^\alpha u = \nu \Delta_x u - \nabla_x p \tag{38}$$

where $0 < \alpha \leq 1$, where the indice t in the fractional derivative $\mathbb{D}^\alpha$ indicates that we derive the field $u(t,x)$ with respect to the time variable t.

For $\alpha = 1$ we recover the classical incompressible Stokes equations. We use the one parameter family in order to suggest that such an equation can be obtain by the fractional embedding of a continuous Lagrangian systems.

Let us consider the continuous Lagrangian function L defined on $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{C}^2$ by

$$L(t,x,y,v,w) = \frac{v^2}{2} - \frac{\nu}{2} w^2 + pw \tag{39}$$

where $\nu \in \mathbb{R}$ and p is a function depending on x.

We have

$$\frac{\partial L}{\partial y} = 0, \ \frac{\partial L}{\partial v} = v, \ \frac{\partial L}{\partial w} = -\nu w + p. \tag{40}$$

Using the fractional embedding we obtain the following results:

Theorem 9. *The $\mathbb{D}_t^{\alpha/2}$ fractional embedding of the continuous Euler-Lagrange equation associated to the Lagrangian density $L(t,x,y,v,w) = \frac{v^2}{2} - \frac{\nu}{2} w^2 + pw$ is given by*

$$\mathbb{D}_t^\alpha u = \nu \Delta_x u - \nabla_x p. \tag{41}$$

Proof: By the fractional least-action principle for fields, the extremals of the fractional functional associated to L are given by

$$-\mathbb{D}_t^{\alpha/2} \left(\frac{\partial L}{\partial v}(z_\alpha(t,x)) \right) - \frac{\partial}{\partial x} \left(\frac{\partial L}{\partial w}(z_\alpha(t,x)) \right) = 0, \tag{42}$$

where $z_\alpha(t,x) = (t, x, u(t,x), d^{\alpha/2} u, \partial_x u)$. As a consequence, we obtain

$$-\mathbb{D}_t^{\alpha/2} \circ \mathbb{D}_t^{\alpha/2} u + \partial_x (\nu \partial_x u - p). \tag{43}$$

As $\mathbb{D}_t^{\alpha/2} \circ \mathbb{D}_t^{\alpha/2} = \mathbb{D}_t^\alpha$ we obtain the result.$\square$

The main consequence of this theorem is the following :

Theorem 10. *If a field $u(t,x)$ is a solution of the incompressible Stokes equation then it is a $\mathbb{D}_t^{1/2}$-weak extremal of the Lagrangian density* $L(t,x,y,v,w) = \frac{v^2}{2} - \frac{\nu}{2}w^2 + pw$.

Proof: This a consequence of theorem 9 and theorem 3 with $\alpha = 1$. $\square$

As a consequence, we have the following diagramm which commutes

$$\begin{tikzcd}
\mathcal{L}_{a,b,\Re} \arrow[r, "\mathrm{Emb}(\mathbb{D}^{1/2})"] \arrow[d, "\mathrm{LAP}"'] & \mathcal{L}^{1/2}_{a,b,\Re} \arrow[d, "\mathrm{WFLAP}"] \\
-\frac{\partial^2 u}{\partial t^2} = \partial_x(\nu \partial_x u - p) \arrow[r, "\mathrm{Emb}(\mathbb{D}^{1/2})"] & \text{Stokes equations}
\end{tikzcd} \tag{44}$$

where LAP and WFLAP corresponds to the least action principle and the weak fractional least action principle respectively.

This diagramm is not correct if we replace the weak fractional least action principle by the fractional least action principle given by theorem 2.

6. Conclusion

The previous results can be used as a conceptual guideline to generalize classical equations of physics in the fractional framework. If the classical equation possesses an additional structure, for example Lagrangian, then we must extend this equation keeping this additional structure, generalized in a natural way. The main remark is that equations by themselves do not have a universal significance, their form depending mostly on the coordinates systems being used to derive them. On the contrary the underlying first principle like the least-action principle carries an information which is of physical interest and not related to the coordinates system which is used. At least, this point of view explains the importance of coherence theorems in all the existing embedding theories of dynamical systems.

From the physical side we can also want to keep homogeneity of the embedded equations whith respect to the fundamental units of Physics. This has been done by Pierre Inizan [13].

Our derivation of a Lagrangian structure for the Stokes equations suggests to look for a Lagrangian structure for the Navier-Stokes or the Euler equations in a unified way, i.e. with a Lagrangian depending on the visocisty as well as the underlying functional set which is related to the choice of the corresponding embedding procedure. It seems that this can not be done in the fractional calculus setting. However, we refer to [4] and [5] where this programm is developped using the non-differentiable and the stochastic embedding theories respectively.

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