Geometry and dynamics of numbers under finite resolution

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Abstract. We define a special set, called resolution space, which corresponds to real numbers obtained via the following resolution rule : every number greater than a given integer a are identified with ∞. This space possesses a natural scaling structure and dynamics. We introduce several notions as locking and transient resonance zones, as well as unstable irrational numbers. This space is the natural object coming in the $1/f$ frequency noise problem. Special numbers as Markoff's irrational are proved to play a specific role. This first criterion must be understood as a finite resolution in space for physical systems.

We then introduce an additional resolution criterion which allows only a finite construction of the previous space. A natural notion of fuzzy zone is defined. This second criterion is interpreted as a finite time experiment in physics.

1 Introduction

Many physical problems involve a careful measure of a given quantity. For example, synchronization of oscillators is studied via a precise measure of the frequency and amplitude of a given signal. Of course, an infinite precision measurement is impossible. This problem has little consequences in general, but as proved in recent works ([2], [11], [12]), some universal phenomenon as 1/f noise come from an explicit dependency on resolution.

The aim of this paper is to provide an understanding of the role of a *finite resolution* assumption on *real* numbers. We define a resolution space, image of an approximation function R_a which takes into account the fact that every number greater than a given integer a are identified with ∞. In other words, we introduce a cutoff which can be understood as a finite resolution in *space*. We then construct iteratively our space via successive applications of two elementary maps on numbers : $1/x$ and $x+1$. We first remark that this construction induces a natural emergence of a *scaling* structure. We distinguish three specific zones : the first one is called *locking* resonance zone. It corresponds to an *attractive* basin of a given rational number. The second one is a *transient* resonance zone. They accumulate on the boundary of locking resonance zones. The third one is the set of *unstable* points which corresponds to irrational numbers with a periodic tail (so quadratic irrational numbers).

It must be pointed out that this is not the *modular group* which induces such a specific structure. Indeed, if we use the modular group in order to construct such a resolution space, then we obtain symmetric locking resonance zones with respect to rational. But, as proved by Planat [11], such configurations are not observed. Instead, we obtain precisely the structure given by our definition of resolution space.

We discuss several particular phenomenon as accumulation of locking resonance zones on irrational numbers of Markoff's type. These numbers play a special role in number theory and diophantine approximation [18].

A convenient representation of the space R_a can be obtained by the following : we represent the approximated exponent of ω versus ω. We obtain a classical object in dynamical system's theory : a *devil's staircase*. We introduce a natural hierarchy of locking resonance zones, following previous works on dynamical devil's staircase [6]. We then try to see how these locking resonance zones are classified. We then see, that when a is not too large, we obtain a classification which is approximately given by the modular group.

We then introduce a second resolution constraint. We allow only a finite number n of inversion during the recursive construction of the space R_a. It can be understood as a constraint on the time of observation. This truncation rule induces *fuzzy* zone around irrational in R_a, where we do not know what exactly happens.

2 Continued fractions, and action of F_2 on Q

We recall in this section classical results dealing with continuous fractions and rational numbers.

To any irreducible fraction p/q of Q we associate the point (p,q) of Z^2 or, equivalently the line (additive subgroup) of Z^2 going through (0,0) and (p,q) of slope q/p of equation $py - qx = 0$. This leads to a bijection between $Q \cup \{\infty\}$ and $P^1(Z)$, the set of vector lines of Z^2, which is usually defined as the set of classes of points of $Z \times Z$ modulo the equivalence $(p,q) \sim (p',q')$ iff there exists $a \in Z$ such that $(p,q) = (ap', aq')$ or $(p',q') = (ap, aq)$.

Each line L in $P^1(Z)$ is isomorphic (as an additive discrete subgroup) to Z , and is generated by one of the two points $(p,q)(-p,-q)$ of L verifying $gcd(p,q) = 1$.

We call such points, irreducible points of Z^2. We denote by I the set of opposites irreducible points pairs $\{(p,q), (-p,-q)\}$ (plotted on fig.2)

We thus have the following one-to-one correspondences :

$$\begin{array}{ccccc} Q \cup \{\infty\} & \longrightarrow & P^1(Z) & \longrightarrow & I \\ \frac{p}{q} & \longmapsto & L : qy - px = 0 & \longmapsto & (p,q) \text{ with } gcd(p,q) = 1. \end{array} \tag{1}$$

The ring of 2×2 Z-matrices, $M_2(Z)$ acts naturally on Z^2. This action induces an action on Q by the so called Möbius transformations as follow :

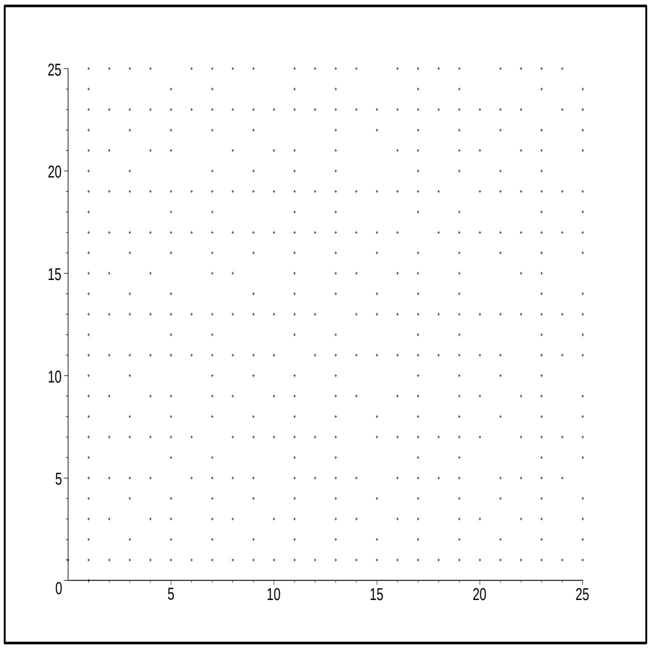

Fig. 1. Irreductible points of Z^2 $(gcd(p,q) = 1)$.

$$
\begin{array}{ccccc}
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : & (p,q) & \longmapsto & (ap+bq, cp+dq) & \\
& \downarrow & & \downarrow & \\
& z = \dfrac{p}{q} & \overset{A}{\longmapsto} & \dfrac{ap+bq}{cp+dq} = \dfrac{az+b}{cz+d}. &
\end{array}
\qquad (2)
$$

It can be shown that A preserves I if and only if $|det(A)| = 1$ or equivalently, A invertible in $M_2(Z)$. Thus, we consider the action of the group $GL_2(Z)$ of invertible 2×2 Z-matrices on Q via Möbius transformations.

Notice that $GL_2(Z)$ contains $PSL_2(Z)$ (determinant one) as a subgroup of index two and also that, since $A \in GL_2(Z)$ preserves I, $gcd(p,q) = 1$ implies $gcd(ap + bq, cp + bq) = 1$ and, thus, the fraction $\dfrac{ap + bq}{cp + dq}$ is irreducible.

2.1 Continued fractions and F_2^+

We recall some basic results on continued fraction. We refer to Khintchine [8] for more details.

Let $(a_0, a_1, \ldots, a_n)$ be a finite sequence of integer with $a_n \neq 0$. We denote by $[a_0, \ldots, a_k]$ the finite continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{}{\ldots + \cfrac{1}{a_k}}}}. \tag{3}$$

In the case of an infinite sequence, we denote also $[a_0, \ldots, a_k, \ldots]$ the associated infinite continued fraction.

Assume that each $a_k > 0$, for $k > 0$, then each irrational has a unique representation as an infinite continued fraction. Nevertheless, each rational has two such representation given by:

$$[a_0, \ldots, a_k] = [a_0, \ldots, a_k - 1, 1] \tag{4}$$

We deduce that each rational x has a unique representation as a finite continued fraction with k odd. By the way, there also exists a unique representation as a continued fraction of even length. Notice that this (odd) continued fraction always gives the irreducible writing of the rational x, and thus the continued fraction represents an irreducible pair of I.

Those continued fractions can be built by the two Möbius transformations,

$$A : x \mapsto x + 1 \text{ and } I : x \mapsto 1/x, \tag{5}$$

applied to the irreducible fraction $1/1$.
The continued fraction $[a_0, \ldots, a_n]$ being written as $A^{a_0} \circ I \circ A^{a_1} \circ \ldots \circ I \circ A^{a_n - 1}(1)$. To work with unimodular (determinant one) transformations, we represent each rational by its odd continued fraction. Indeed, $det(A) = 1$ and $det(I) = -1$, and the odd representation has an even number of inversions. Regrouping inversion two by two, and introducing the tranformation $B = I \circ A \circ I$, we obtain:

Lemma 1. *Any positive rational number admits a unique writing:*

$$[a_0, \ldots, a_{2n}] = A^{a_0} \circ B^{a_1} \circ \ldots \circ A^{a_{2n} - 1}(1). \tag{6}$$

The matrices of A and B are:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \tag{7}$$

They generates the modular group $PSL_2(Z)$ [19] of unimodular Möbius transformations which is isomorphic to the free group of rank two F_2, A and B being two free generators. Denoting F_2^+ the semi-group of words written with positives powers of A and B, we have the following

Corollary 1. *The application:*

$$\begin{array}{ccc} Q^{+*} & \simeq & F_2^+ \\ [a_0, \ldots, a_{2n}] & \longmapsto & A^{a_0} \circ B^{a_1} \circ \ldots \circ A^{a_{2n}-1} \end{array} \tag{8}$$

is a bijection.
Moreover F_2^+ acts on Q by left multiplication.

This is a simple reformulating of the previous lemma. In particular, this establishes a bijection between N^*, the set of positive numbers, with the set of words A^i with $i \geq 0$.

2.2 Geometry of continued fractions

The words B^i, with $i \geq 0$, applied on 1, generate the fractions $1/n$ with $n \in N^*$. Going back to the representation of irreducible fractions by points (p, q) of Z^2, with $gcd(p, q) = 1$, we represent these two sets as two affine half lines, L_∞ and L_0 coming out of $(1, 1)$, and of respective slope 0 and ∞.

We then let F_2^+ (it's a linear group) act on Z^2, and observe the images of those two half-lines. We obtain the following figure 2.2.

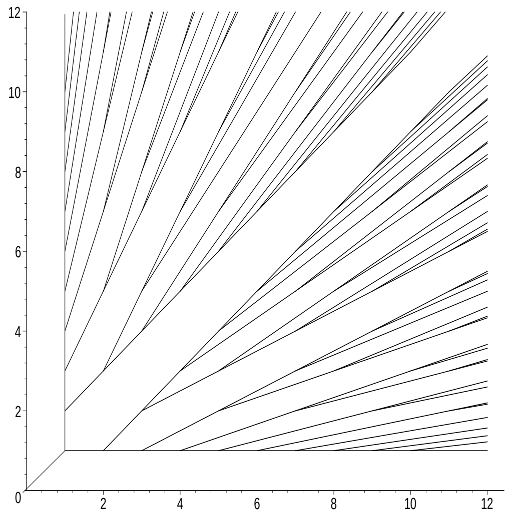

Fig. 2. A plane Farey tree

The striking facts are that it's a binary tree, $\mathcal{T}$, that it's nodes are exactly the points (p, q) with $gcd(p, q) = 1$, naturally associated to the irreducible factions p/q, and therefore indexed by Q and moreover, that each point (except $(1,1)$), is both living on a straight branch (called cross branch) and the root point of a new springing straight branch. Moreover, each of those branches is exactly the image, by a uniquely determined word of F_2^+, of one of the two half lines L_0 and L_∞. Leaving aside the proofs of those facts, we just give the following properties, dealing with continuous fractions :

Lemma 2. *If $\frac{p}{q} \neq 1$ is a node of $\mathcal{T}$, and $[a_0, a_1, \ldots, a_{2n}]$ its odd continuous fraction.*

- The root of its cross branch is: $[a_0, a_1, \ldots, a_{2n-1}, 1]$ if $a_{2n} > 1$, $[a_0, a_1, \ldots, a_{2n-2} + 1]$ if $a_{2n} = 1$.

- The slope of this cross branch is $[0, a_0, a_1, \ldots, a_{2n-1} - 1, 1] = [0, a_0, a_1, \ldots, a_{2n-1}]$ if $a_{2n} > 1$, $[0, a_0, a_1, \ldots, a_{2n-2}]$ if $a_{2n} = 1$.

- As a consequence, the slope of (p,q)'s springing branch is $[a_0, a_1, \ldots, a_{2n} - 1]$ if $a_{2n} > 1$.

Proof. We admitted that the cross branch of (p,q), call it C, is exactly the image of L_0 or L_∞ by a word of F_2^+.

We suppose that C is the image of L_∞, the horizontal half line, by the word M. The root of C is then M(1), and p/q is $M(l) = M \circ A^{l-1}(1)$ with $l \in L_0 \simeq N$ (and $l > 1$ since (p,q) is not the root of C).

First, M doesn't end with an A. If it would, call N it's subword obtained by cutting the last A. $N(1)$ would belong to C and M(1) wouldn't be the root.

Second, $l = a_n$. Indeed, $p/q = M \circ A^{l-1}(1) = A^{a_0} \circ B^{a_1} \circ \ldots \circ A^{a_{2n}-1}(1)$, and, with 1 we obtain $l = a_n$.

As a consequence $M = A^{a_0} \circ B^{a_1} \circ \ldots \circ B^{a_{2n-1}}$ and $M(1) = A^{a_0} \circ B^{a_1} \circ \ldots \circ B^{a_{2n-1}} \circ A^0(1) = [a_0, a_1, \ldots, a_{2n-1}, 1]$ which establishes the first point.

The case C is the image of L_0 by M is treated the same way, replacing B by A. (and M ends with a B).

We skip to the second point which becomes clearer if we notice that L_∞ is parallel to the horizontal axe, going through $(0,0)$ and $(1,0) = B^{-1}((1,1))$. $(0,0)$ is fixed by M (it's linear), and $M((1,0)) = M \circ B^{-1}((1,1)) = A^{a_0} \circ B^{a_1} \circ \ldots \circ B^{a_{2n-1}-1}(1,1)$.

Therefore, the slope of the image of the horizontal line (and thus of B) is the rational associated to $M((1,0))$ which is $[a_0, a_1, \ldots, a_{2n-1} - 1, 1]$ as announced.

For the third point, just think the springing branch as a cross branch of an appropriated rationnal. Notice that this branch will then be an image of L_0, and the same proof should be done, replacing B by A and $(1,0)$ by $(0,1)$.

This tree which is inspired by the Farey tree on the hyperbolic disk gives us a good description of Q, for continuous fractions calculus (but very complex if we wish to translate, in this language, multiplication by 2!). The first point of the last lemma, for example, shows that we can obtain the successive truncated fractions of a gives one, just by sliding down the tree from springing point to another.

The two next ones gives us another information: take a rational $p/q = [a_0, a_1, \ldots, a_{2n}]$, and his two sons, going up in the tree: $x_1 = [a_0, a_1, \ldots, a_{2n}+1]$ and $x_2 = [a_0, a_1, \ldots, a_{2n} - 1, 1, 1]$. The two springing branches of those points are then parallels of slope q/p. Those branches (with the two segments $[p/q, x_1]$ and $[p/q, x_2]$ delimit a domain $D(p/q)$ of the plane which will never be crossed by any other branch of the tree.

Theses domains pave the plane. (look fig2.2). The only vector line out of (0,0) which, once entered, never leaves it, is the line $L(p/q) : py - qx = 0$, associated to p/q as previously. Thus the only point of $P_1(Z) = Q$ "contained" in $D(p/q)$ is p/q, and we can say, in this sense, that it's wideness is 0. We'll see in the next section, that those domains open when we restrict the set of available words of F_2^+, truncating the tree.

3 Arithmetic resolution space.

The upper construction gives a description of $R^+ \cup \{\infty\}$, with infinite resolution. The rational being associated to finite words of F_2^+, and irrational (and ∞) to infinite words. We now give a way to introduce finite resolution, keeping consistency with continuous fraction. This question arises naturally in some physical problems.

Let a be a positive integer. We assume that the following finite resolution criterion is satisfied : all real numbers $x \geq a$ is identified with ∞. This means that the only words of F_2^+, of type A^n, we tolerate are the words A^i with $i < a$. The other words being mapped to the limit word A^∞.

Impose the same resolution condition near 0, mapping the words B^i to B^∞, for any $i > a$. We want those resolution conditions to be imposed near any rational of Q^+. Let $F_2^+(a)$ be the set of words of F_2^+ containing no subword A^i or B^i with $i > a$, and define the map :

$$\begin{aligned} R_a : R^+ \cup \{\infty\} &\longrightarrow R^+ \cup \{\infty\} \\ M(1) &\longmapsto M_a(1) \end{aligned} \tag{9}$$

where M_a is the word obtained from M, by replacing in M the first word A^i or B^i with $i > a$ by, respectively A^∞ and B^∞. Notice many words are fixed by R_a, as for example, the word infinite $ABABA$.... The tree corresponding to $F_2^+(a)$ is given in fig 3. Observe that the domains of any node of this tree (in the sense defined in previous section) are more open and the numbers associated to the vector lines "contained" in it are sent, by R_a to the rational corresponding to the node.

The approximation function R_a defined above has a rather complicated structure with many autosimilarities. It leads, by iterations, to a dynamical system on numbers. R_a is represented below (Fig. 4).

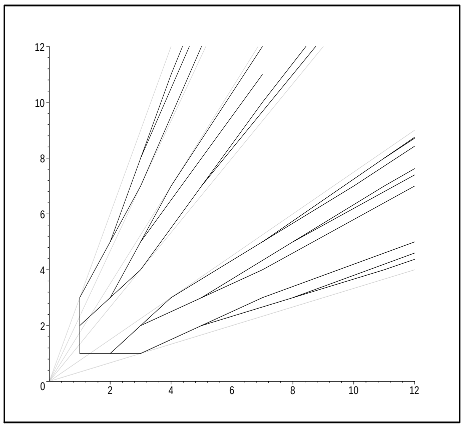

Fig. 3. The tree truncated, for a=2

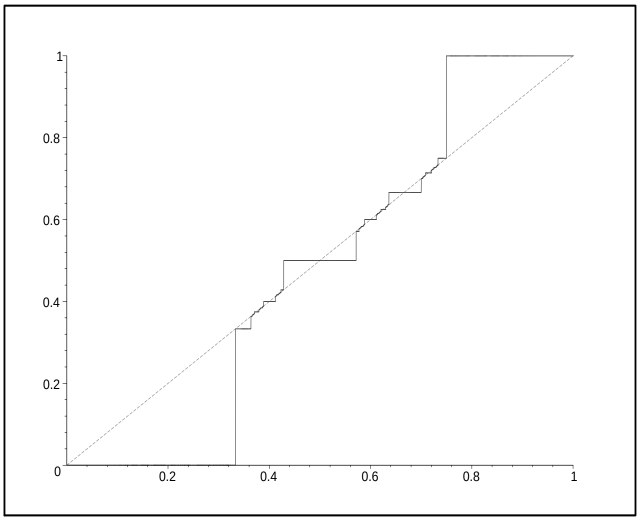

Fig. 4. Approximation function (for $a = 3$).

We can characterize the different horizontal landings of this function:
locking resonance zones, which are basin of attraction where all real numbers are approximated by the rational defining this zone, itself being fixed (locked) by R_a;
transient resonance zones which accumulate on each side of any locking zone, but only possess a semi basin of attraction (we say also that they are semi-stable). The numbers in these transient zones (including p/q) only reach their locking rational after several iteration of R_a.

These landings can be built on an iterative way, as images of the two fundamental attractive zones: $[a, \infty[\rightarrow \infty$ and $]0, 1/a] \rightarrow 0$, by the maps of $F_2^+(a)$, written as iterations of A and B.
Recall $A : x \mapsto x+1$ and $B : x \mapsto 1/x \mapsto 1+1/x \mapsto 1/(1+1/x) = x/(x+1)$.

For example $A(]0, 1/a] \rightarrow 0)$ gives $]1, 1+1/a] \rightarrow 1$ and $B([a, \infty[\rightarrow \infty)$ gives $[a/(a+1) = 1 - 1/(a+1), 1[\rightarrow 1$, which builds the locking basin of 1. The border of these zone will later be called $\nu^+(1)$ and $\nu^-(1)$. An analogous construction of the transient zones on the left the locking zone of 1 is given later in section 4.2:

By iterating the previous maps (at most a consecutive A and B are tolerated), these dynamics build the resolution function R_a of fig4. The two types of dynamics (transient and locking) can also be observed by a close look at the function error of resolution: $|R_a(x) - x|$ given on Fig.5.
We define the resolution space in a, to be the image of $R^+ \cup \infty$ by R_a.

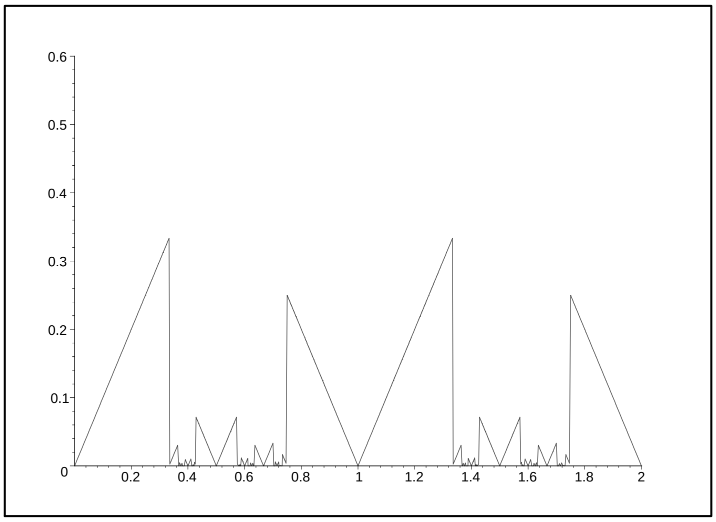

Fig. 5. Resolution error, $a = 3$.

We can now state the following theorem :

Theorem 1. *Let $a \in \mathbf{N}$ be given, the resolution set $Im(R_a)$ has the following structure :*

i) locking rational p/q: corresponding to a locking interval, with rational boundaries $\nu^+(p/q)$ and $\nu^-(p/q)$ (with $p/q \in]\nu^-(p/q), \nu^+(p/q)[$) such that for all $x \in [\nu^-(p/q), \nu^+(p/q)]$, the resolution of x is $R_a(x) = p/q$.

ii) transient rational p/q: corresponding to a transient interval with boundaries $]\nu(p/q), p/q[$ such that, for all $x \in]p/q, \nu(p/q)]$, the resolution of x is $R_a(x) = p/q$. (and its not the case for any $x \in]0, p/q]$

iii) irrational obtained as accumulation of locking resonance zones.

iv) irrational obtained as accumulation of transient and locking resonance zones.

As already pointed out in the introduction, this set occurs in 1/f frequency noise. Next section deals with some particular properties of this space.

3.1 An analogue of the dynamical devil's staircase in number theory.

The devil's staircase is a common object in dynamical system theory. Briefly, synchronization can be studied via one parameter family of diffeomorphism of the circle [13]. One can take the Arnold's model [3] defined by $f_\omega(\theta) = \theta + \omega + \epsilon \sin(\theta)$ mod. 2π for $\theta \in S^1$. When $\epsilon = 0$, we have a *rigid* rotation of frequency ω. When $\epsilon \neq 0$, we can study the behavior of f_ω via the *rotation number* $\rho(\omega)$, which corresponds to a *mean* frequency associated to the system (see [5], [6]). Periodic behaviors of the system are given by *rational* rotation numbers. One can prove that, each rational number p/q gives rise to an interval in ω of constant rotation number, of size ϵ^q (see [4], [6]). If we picture the value of the rotation number versus ω for a fixed value of ϵ, then we obtain a Devil's staircase [6].

In this case, a natural hierarchy of zones in ω of constant rational frequency numbers is given by *Farey's* numbers [6]. Indeed, as the length in ω of the p/q zone is of order ϵ^q, these zones are classified following Farey's hierarchy.

What's about our devil's staircase ? First, when a is not to large, we obtain for locking zones of resonance in the unit interval $[0, 1]$, a hierarchy (with respect to the length of these zones) which can be approximately described via the modular group. The typical value of a where we can take the modular group to classify locking resonance zones is $a = 4$. When a is greater than 4, locking resonance zones are *not* at all classified by the modular group.

We note that the previous observation can be related to Planat [13] remark that when a is too large, the Riemann hypothesis approach to 1/f noise theory is no longer valid. This can be interpreted as a translation of the modular group deficiency to classify locking resonance zones (see also the end of this article).

We emphasize that the resolution space corresponding to physical and experimental results about 1/f noise is generated by our approach. This means that the *modular* group is not the natural object associated to resolution. Elementaries maps as T and I do not define a group at all. Resolution induces specific indempotent properties like $T^p = T^a$ for all $p \geq a$.

4 Behavior of continued fractions with bounded partial quotients.

In this section we characterize the various dynamical behaviors of numbers, under iterations of R_a, using their continued fractions.

4.1 On the boundary of locking resonances.

Let $x = p/q$ be a locked number. Following Planat [11], we denote by $\nu^+(x)$ (resp. $\nu^-(x)$), the rational number which defines the right (resp. left) boundary of the locking zone of x. We have the following elementary properties :

Lemma 3. *For any locked number, we have*

$$\begin{array}{ll} \nu^\sigma(x+1) = 1 + \nu^\sigma(x), & \sigma = \pm, \\ \nu^\sigma(1/x) \quad = 1/\nu^{-\sigma}(x), & \end{array} \tag{10}$$

These functional equations allow us to define ν^σ, $\sigma = \pm$ for all rational numbers.

Lemma 4. *The zone of locking resonance for a given rational*

$$\frac{p}{q} = [a_0, a_1, \ldots, a_k],$$

is determined by the following boundary numbers

$$\begin{array}{l} \nu = [a_0, \ldots, a_k, a], \\ \beta = [a_0, \ldots, a_k - 1, 1, a]. \end{array} \tag{11}$$

Proof. We have

$$\nu^+([a_0, \ldots, a_k]) = \nu^+(a_0 + \frac{1}{[a_1, \ldots, a_k]}) = a_0 + \nu^+(\frac{1}{[a_1, \ldots, a_k]}). \tag{12}$$

Then,

$$\nu^+([a_0, \ldots, a_k]) = a_0 + \frac{1}{\nu^-([a_1, \ldots, a_k])}, \tag{13}$$

and, by induction we obtain

$$\nu^+([a_0, \ldots, a_k]) = [a_0, \ldots, a_{k-1}, \nu^\sigma(a_k)], \tag{14}$$

where $\sigma = +$ if k is odd and $-$ otherwise.

As $\nu^+(a_k) = a_k + \dfrac{1}{a}$, and $\nu^-(a_k) = a_k - 1 + \dfrac{1}{1 + \dfrac{1}{a}}$, we obtain either ν or β for $\nu^\sigma([a_0, \ldots, a_k])$, $\sigma = \pm$.

This result deals with *diophantine approximation* via classical relation between continued fractions and best approximation of real numbers. We point out that from the physical view point, continued fraction and resolution arguments come from diophantine approximation problems (see [8]).

4.2 Dynamics of transient resonance zones.

$BA^{a-1}([1-1/(a+1),1[\to 1) = B([a-1+a/(a+1),a[\to a) = [B(a-1+a/(a+1)),a/(a+1)[\to a/(a+1)$, which sticks on the left of $[a/(a+1) = 1-1/(a+1),1[\to 1$ attractive zone.

Transit resonance zones can be obtained via a simple recursive formula. We first construct the *primary* transient resonance zone, given by the partial locking resonance zone: take the locking interval $[a/a+1,1]$, and make $x \mapsto x+a-1$.

We obtain the interval $[a-1/a+1,a]$. By taking the image under $x \mapsto 1/x$ of this interval, we obtain the primary resonance zone.

We then produce an infinite sequence of transient resonance zones by applying the following rule on each generation of transient resonances : do $x \mapsto x+1$, $x \mapsto 1/x$, $x \mapsto x+a-1$ and $x \mapsto 1/x$ (see fig.4.2).

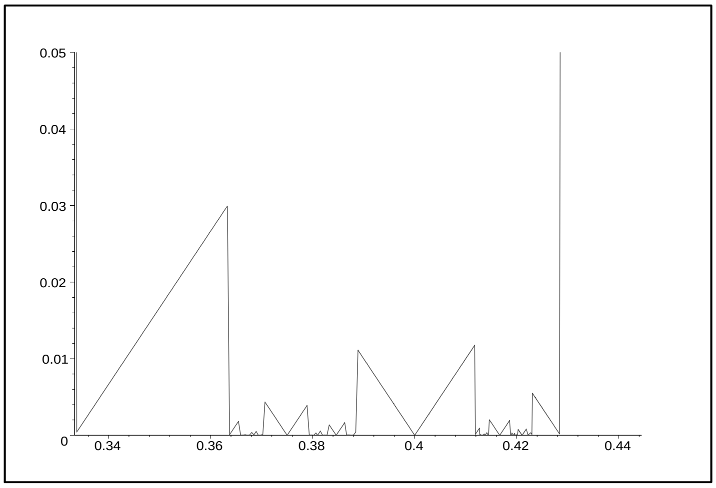

Fig. 6. Accumulation of locking and transient resonances, near $\sqrt{3}-1$ for $a=3$.

Then, we must iterate the following map

$$P = \begin{pmatrix} 1 & 1 \\ a-1 & a \end{pmatrix}, \tag{15}$$

on the primary transient resonance.

We can easily prove that this map is hyperbolic. By iteration, it converges toward a fixed point of P^n. Precisely, we have

Lemma 5. *By iteration of P we obtain an irrational number such that its tails , at a given depth, becomes periodic of period 2, given by*

$$[\dots,1,a-1,1,a-1,1,a-1,1,a-1,\dots]. \tag{16}$$

We then obtain a *quadratic* irrational number.

These points are called *unstable*. Let x be a fixed point of P (i.e. $P(x) = x$). Then, x is solution of $(a-1)x^2+(a-1)x-1=0$. For example, for $a=2$, we obtain $x = \dfrac{-1+\sqrt{5}}{2}$ which is the golden number. The stability of this fixed point is given by the sign of $P'(x) = \dfrac{1}{((a-1)x+a)^2}$. As, $P'(x) > 0$, this point is unstable from the dynamical viewpoint.

Assume that we take a small random perturbation of this unstable point. Then, we can go, with equal chance, toward two different locking zone of resonance. Then, if a system depends on such a number, we will have a high sensibility to small perturbation in this zone, which is not the case for locking or transient zone of resonances.

4.3 Accumulation of locking resonance zones.

With respect to the previous lemma on the behavior of transient resonance zones under iteration of the map P, we can try to do the same for locking zone of resonances.

The basic idea is to look for a recursive construction of the rational defining locking resonance zones. We restrict ourselves to look for these rational in the interval $[0,1]$.

At the first step, they are obtain by $1/x$ from the integer $2,3,\ldots,a-1$. We then obtain at step one the following set of primary resonances

$$R_1 = \{1/(a-1),\ldots,1/3,1/2\}. \tag{17}$$

To obtain the second generation of locking resonances, we must do a translation of the form $x \mapsto x+t$ where $1 \leq t \leq a-1$. Then, we take $1/x$. The elementary map is defined by

$$A_t = \begin{pmatrix} 0 & 1 \\ 1 & t \end{pmatrix}. \tag{18}$$

Let us first see what happens when we iterate a given number from R_1 by A_t, which of course is the most elementary kind of resonances that we can produce from R_1 (as one must, in general, allow different combination of A_t in the iterate). We then have the following accumulation lemma.

Lemma 6. (accumulation lemma) *Any rational* $1/x$, $2 \geq x \leq a-1$ *of* R_1 *converges toward an irrational number of the following form*

$$\bar{t} = [t,t,\ldots,t,\ldots]. \tag{19}$$

At each generation of resonances, the action of A_t put a new t in the sequence defining the continued fraction of the previous generation step.

For example, take $1/x$, which can be written $[0,x]$. By A_1, we obtain $1/(1+1/x)$ which takes the form $[0,1,x]$. By induction, the i-generation of resonances is given $[0,1,\ldots,1,x]$ with i times 1 in the continued fraction. Then, when $i\rightarrow\infty$, we obtain an irrational number of the form given in the lemma. This irrational number is the *golden* number. Precisely, the golden number is the best approximated irrational number by rational. It is defined by $\dfrac{-1+\sqrt{5}}{2}$ and its continued fraction expansion is given by $[1,\ldots,1,\ldots]$. So, by induction of A_1, we obtain the set of poorly approximated irrational number by rational (in the continued fraction sense).

Let x be an irrational as in the lemma. It is a fixed point of A_t. Then, it is a solution of $x^2+tx-1=0$, which are given by $x=\dfrac{-t+\sqrt{t^2+4}}{2}$. The stability of this point is given by the sign of $A_t'(x)$. As $A_t'(x)=-x^2$, this point is stable.

The previous lemma can be generalized for other inductive scheme. We must allow combination like $A_1\circ A_{a-1}\circ A_2$ etc., in order to obtain a complete description of the accumulation limit set. Nevertheless, no new phenomenon occurs for these kind of combination, and they are all, more or less, relevant of the "basic" accumulation lemma.

4.4 Approximation property.

We shall say that ξ is approximable by rational to order n if there is a $K(\xi)$, depending only on ξ, for which

$$\left|\frac{p}{q}-\xi\right|<\frac{K(\xi)}{q^n}, \tag{20}$$

has an infinity of solutions.

A rational is approximable to order 1 and to no higher order. A quadratic irrational is approximable to order 2 and to no higher order.

Theorem 2. *An irrational real number with bounded partial quotients is approximable to order 2 and to no higher order.*

A conjecture asserts that : an irrational real number with bounded partial quotient is either quadratic or transcendental. We refer to [1] for more details and to [14] for an example of transcendental number with bounded partial quotient.

By theorem2, two irrational real numbers with bounded partial quotients are distinguished by there constant of approximation, $K(\xi)$. It is possible to describe the spectrum of this constant. We obtain the so called Markoff's spectrum. We refer to [18] and [16] for more details.

4.5 Lenght of locking resonance zone.

The previous section allow us to compute the size of the locking resonance zone for each rational. We do this by induction on the generation of locking resonant rational. We will see, in the last section, that these length play an important role in physics for the understanding of 1/f noise fluctuation.

We define the *length* function as

$$l(x) = \nu^+(x) - \nu^-(x). \tag{21}$$

We have

Lemma 7. *The length function satisfies*

$$\begin{aligned} l(x+1) &= l(x), \\ l(1/x) &= \frac{l(x)}{\nu^+(x)\nu^-(x)}. \end{aligned} \tag{22}$$

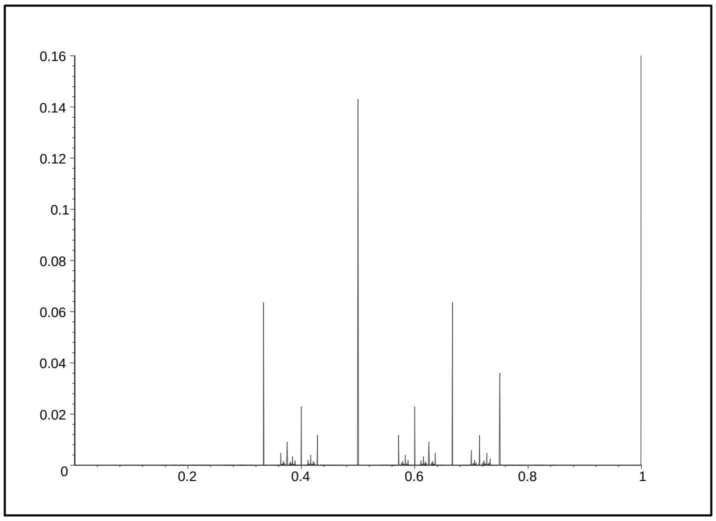

Fig. 7. Lenght of locking resonance zone

The preceding picture leads us to the following problem : is this figure symmetric with respect to $x = 1/2$? If you look at this picture you can see on the right a little part which has no symmetric counterpart. Nevertheless, all the rest seems symmetric. The following lemma gives a complete description of this phenomenon.

First, we prove that the length function is symmetric with respect to $x = 1/2$.

Lemma 8. *For $x \in [2, \infty]$, we have*

$$l\left(\frac{x-1}{x}\right) = l\left(\frac{1}{x}\right). \tag{23}$$

Proof. For all $x \in [1, \infty[$, we have via lemma 3, 7

$$\begin{aligned} \nu^\sigma(x-1) &= \nu^\sigma(x) - 1, \\ l(x-1) &= l(x). \end{aligned} \tag{24}$$

We note that $\dfrac{x}{x-1} = 1 + \dfrac{1}{x-1}$, then

$$l\left(\frac{x-1}{x}\right) = \frac{l(x/x-1)}{\nu^+(x/x-1)\nu^-(x/x-1)} = \frac{l(1+(1/x-1))}{\nu^+(1+(1/x-1))\nu^-(1+(1/x-1))}. \tag{25}$$

We obtain

$$\begin{aligned} l(x-1/x) &= l(1/x-1)\frac{1}{\nu^+(1/x-1)\nu^-(1/x-1)} \\ &= l(x-1)\frac{1}{\nu^+(x-1)\nu^-(x-1)}\frac{1}{\nu^+(1/x-1)\nu^-(1/x-1)}, \end{aligned} \tag{26}$$

and finally,

$$l(x-1/x) = l(x-1)\frac{1}{(\nu^+(x-1)+1)(\nu^-(x-1)+1)} = \frac{l(x)}{\nu^+(x)\nu^-(x)} = l(1/x). \tag{27}$$

So the length function is symmetric with respect to $x = 1/2$. What's about the disymmetry of the picture ?

First, we state a technical lemma, which gives the continued fraction expansion of $1-x$ from x's one.

Lemma 9. *If $x = [0, a_1, a_2, \ldots]$ with $a_1 \geq 2$, then we have $1 - x = [0, 1, a_1 - 1, a_2, \ldots]$.*

Proof. If $x = [0, a_1, a_2, \ldots]$, then $1/x = [a_1, a_2, \ldots]$. We note that $\dfrac{1}{1-x} = 1 + \dfrac{1}{\frac{1}{x} - 1}$. Then, we obtain

$$\frac{1}{1-x} = [1, a_1 - 1, a_2, \ldots]. \tag{28}$$

Hence, by inversion, we have

$$1 - x = [0, 1, a_1 - 1, a_2, \ldots]. \tag{29}$$

We are now able to specify points of $[1/2, 1]$ with no symmetric. We have

Lemma 10. *If $x \in [1/2, 1]$ with its continued fraction expansion of the form $x = [0, 1, a_1, a_2, \ldots]$, with $a_1 = a_{\max} - 1$, then $1 - x$ belongs to the transient zone of 0.*

Proof. If $a_1 = a_{max} - 1$, then $1 - x = [0, a_{\max}, a_2, \ldots]$ and by the truncation rule, this point is identified with 0. Then, x, which possess a locking zone of resonance, has no symmetric in $R_{a_{\max}}$.

By lemma 10, all the locked rational in the interval defined by $[0, 1, a_{\max} - 1]$ and $[0, 1, a_{\max} - 1, 1, a_{\max} - 1, \ldots$ possess no symmetric counterpart. This explain the existence of a single zone on the right which broke the symmetry.

We emphasize that this property reflects the fact that the modular group is not adapted to classify locking zone of resonances.

5 Truncated resolution space.

Despite our first resolution constraint, our space contains irrational numbers. From a physical viewpoint, this is not at all realistic, and one need to introduce a second natural constraint in order to provide a space which can really occurred in physics.

The basic idea is the following. When we construct our resolution space by induction, we have a natural notion of *generation* of resonances (as used for example in the section about accumulation of locking resonance zone). More or less, this notion of generation counts the number n of invertion that we allow for the construction of locking resonance zone. From the physical view point, this must be understood as a finite time observation. Then truncation in a is a *space* resolution and truncation in n is a *time* resolution.

Let us consider an integer n. We restrict ourselves to the structure of the resolution space on $[0, 1]$.

Definition 1. The *truncated resolution space* of order n, denoted by R_a^n, is the set of rational number in $[0, 1]$ obtain by iteration of $A : x \mapsto x + 1$ and $I : x \mapsto 1/x$ on the set $R_1 = \{1/(a-1), \ldots, 1/2\}$ with only n iteration of I. Precisely, for each rational of R_1 we allow map of the form

$$I \circ A^{s_1} \circ I \circ A^{s_2} \circ I \circ A^{s_3} \circ I \circ \ldots \circ A^{s_n}, \tag{30}$$

with $1 \leq s_i \leq a - 1$.

We then obtain the following picture for $a = 3$ and $n = 2$.

So, there exists *fuzzy* zone between locking and transient resonances zone in which we do not know what happens really, as we have not get any information on this part via our construction. Finite resolution in time induces natural fuzzy zone where one can not say what the system really do.

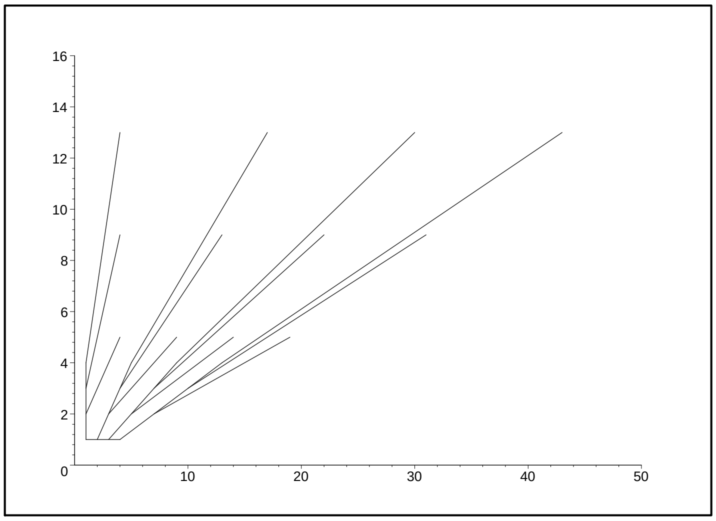

Fig. 8. Truncated resolution space.

References

1. Allouche, J-P, Algebraic and analytic randomness, this volume
2. Nottale, L, Scale-relativity and quantization of the universe I. Theoretical framework, Astron. Astrophys. 327, 867-889, (1997).
3. Arnold, V,I, Small denominators I: On the mapping of a circle onto itself, Trans. Amer. Math. Soc. Ser. 2, 46 (1965) 213-284.
4. Arnold, V,I, Remarks on the perturbation theory for problems of Mathieu type, Russian Math. Surveys 38:4 (1983), 215-233.
5. Yoccoz, J-C, An introduction to small divisors problems, in "*From number theory to physics*", Springer-Verlag, New-York, 659-679, (1992).
6. Cvitanovik, P, Circle maps : irrational winding, in "*From number theory to physics*", Springer-Verlag, New-York, (1992).
7. Hardy, G, Wright, E, An introduction to the theory of numbers, Oxford University Press, Amen House, London E.C.4, (1965).
8. Kintchine, A. Ya., Continued fractions, P. Noordhoff Ltd., Groningen 1963.
9. G.A. Jones, D. Singerman, K. Wicks, The modular group and generalized Farey graphs, in Groups St Andrews, 1989, London Math. Soc. Lectures Notes 160, 316-338, (1991).
10. G. Jones, D. Singerman, Maps, hypermaps and triangle groups, in The Grothendieck theory of Dessins d'enfants, London. Math. Soc. Lectures notes series 200, (1994).
11. Planat, M, Dos Santos, S, Ratier, N, Cresson, J, Perrine, S, Close to resonance interaction of radiofrequency wawes in a Schottky diode mixer: 1/f noise and number theory, in "*Quantum 1/f noise and other low frequency fluctuations in electronic devices*", edited by A. Chung and P. Handel, AIP Press (1999).
12. Planat, M, Dos Santos, S, Cresson, J, Perrine, S, 1/f frequency noise in a communication receiver and the Riemann hypothesis, Hong Kong, (1999).

13. Planat, M, 1/f frequency noise in a communication receiver and the Riemann hypothesis, this volume.
14. Queffélec, Transcendance des fractions continues de Thue-Morse, Jounal of Number theory 73, 201-211, (1998).
15. Serre, J-P, A course in Arithmetic, Graduate text in Mathematics 7, Springer-Verlag, New-York-Heidelberg-Berlin, 1973.
16. Series, C, The geometry of Markoffoff numbers, The Mathematical Intelligencer Vol.7,no.3,20-29, (1985).
17. Apostol, T, M, Modular functions and Dirichlet series in number theory, Graduate Text in Mathematics 41, Springer-Verlag, 1976.
18. Perrine, S, Generalized Markoff theories and the associated conformal geometries, this volume.
19. Cohen, P, On the modular function and its importance in arithmetic, this volume
20. Haas, A, Diophantine approximation on hyperbolic Riemann surfaces, Acta Mathematica 156, 33-82, (1986).